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Mineralogical Society.

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1891.

EXCERPT MINUTES OF COUNCIL MEETING.

Held in the Apartments of the Geological Society, April 14th, 1891.

R. H. Scott, Esq., M.A., F.R.S., President, in the Chair.

r. John William Evans, F.G.S., and Mr. Alfred E. Tutton, F.C.S., elected Members.

hanks were voted to Mr. Solly for the gift of three figures, and to Mr. s for the loan of three blocks, for use in the illustration of their re- tive papers in No. 43 of the Journal.

GENERAL MEETING.

eld in the Apartments of the Geological Society, April 14th, 1891.

R. H. Scott, Esq., M.A., F.R.S., President, in the Chair.

irteen Members and two Visitors were present.

e following papers were read :—

On the Occurrence of an aluminous Serpentine (Pseudophyte) with ilike appearance near Kynance Cave." By Howard Fox, Esq., F.G.S. mmunicated by Mr. Teall.) (page 275.)

Note on the Occurrence of Melanterite in the Upper Eocene Strata of Thames Basin." By the Rev. A. Irving, B.A., D.Sc. (Communi- l by the Secretary.) (page 392.)

Notes on various American Minerals, including Bastnäsite, Oligoclase, tz, Sapphire, and a Meteoric Iron." By G. F. Kunz, Esq. (page 394.)

On the Devitrification of Cracked and Brecciated Obsidian." By essor Grenville A. J. Cole, F.G.S. (page 272.)

Notes on Crystallites." By F. Rutley, Esq., F.G.S. (page 261.)

On the Method of Observing the Absorption-spectra of Minerals with id of the Microscope." By Allan Dick, Esq. (Communicated by ecretary.)

EXCERPT MINUTES OF COUNCIL MEETING.

Held in the Apartments of the Geological Society, June 16th, 1891.

R. H. Scott, Esq., M.A., F.R.S., President, in the Chair.

The following Members were selected for nomination as Officers and Members of Council at the Annual Meeting :—

President. Prof. N. S. Maskelyne, M.P., F.R.S. (in place of Mr. Scott)

Treasurer. Mr. F. W. Rudler, F.G.S. (in place of Professor Bonney.)

Council. Rev. H. P. Gurney, M.A., F.G.S.
Prof. G. D. Liveing, M.A., F.R.S.
Maj.-Gen. C. A. McMahon, F.G.S.
Mr. J. J. H. Teall, M.A., F.R.S. (in place of the retiring members Dr. Burghardt, Mr. Miers, Mr. Solly, and Prof. J. M. Thoms

GENERAL MEETING.

Held in the Apartments of the Geological Society, June 16th, 1891.

R. H. Scott, Esq., M.A., F.R.S., President, in the Chair.

Eight members were present.

The following papers were read :—

"On the Occurrence of Sapphire in Scotland." By Prof. M. Forster Heddle, M.D., F.R.S.E. (Page 389.)

"On the Optical Properties of Gyrolite." By the same. (Page 3

"On the Optical Indicatrix and the Transmission of Light in Crysta By L. Fletcher, M.A., F.R.S. (Page 278).

A circular of which the following is a translation has been received Dr. Aristides Brezina. Director of the Mineral Department, Na History Museum at Vienna.

On the 26th of September a young man was captured in the Mi logical Department of the Natural History Museum at Vienna, who managed to conceal himself in the parapet of a window shortly befor museum was closed ; he was found to have upon him a loaded revo

ther with files and other burglarious implements; in his possession also found a collection of about 600 gems, partly cut and partly rough es.

he thief, who had a passport from Berlin under the name of Hugo n, and also gave the names of Krony, Kronek, Karnak, Kronicsalsky, t years old, 5 ft. 7 in. in height, thin, with somewhat long handsome ile, brown complexion, dark hair, gray eyes, thin brown beard, alto- er superior appearance; he has made several journeys in Germany, nce, Switzerland, and Italy. Between the middle of July and beginning eptember he has passed through Pyrmont, Ems, Strassburg, Basel, n, Genoa, Nice, Monaco, Genoa and Venice, to Vienna, and claims ave bought the majority of the stones (of which he does not know the es with the exception of the Rock Crystal) from a barber in Marseilles. would be desirable to discover the former possessor of all or some of e stones, and the following list is given, with the request for informa- regarding their previous habitat.

uclase: 1 crystal, 2 cm. in length; 1 cleavage fragment 1 cm. ss.

merald: 6 small crystals from Muso; 1 moderate-sized crystal of ior colour with calcite from Muso; 1 large Russian crystal.

eryl: 3 large aqua marines, partly cut; 1 green crystal with polished ; 1 yellow crystal; 3 small crystals; 1 aqua-marine cut as a iant.

rysoberyl: 6 small catseyes; 3 crystals.

pinel; indigo blue, cut as a brilliant; about 30 small crystals from on, 1 small crystal on the matrix from North America.

ar Sapphire, cut en cabochon.

ppaz: 2 rose-red crystals; 36 Schneckenstein topazes; 4 large yellow ilian topazes; 4 water-worn crystals; 7 crystals from different ities.

rysolite: 7 crystals, 2 cut as brilliants.

rcon: 1 from Canada; about 50 small zircons from Frederiksvärn Miask.

urmaline: 1 crystal from Prävali; 4 from Elba; 1 small clear crystal black termination; 50 black crystals; 1 clear green from Binnen-

ocrase: 4 from Ala; 1 from Vesuvius.

inite: 3 from Dauphiné; 1 from Scopi.

rnet: Dodecahedron with icositetrahedron; about 40 Hessonite les.

Quartz : 300 Marmora diamonds ; 3 Flêches d'Amour, polished
Carrara crystals ; 1 complete crystal with curved edges ; 4 gray to bl
crystals ; 10 crystals (various) ; 6 polished fossil woods.

Opal : 1 fire-opal ; 3 milk-opals ; 2 polished wood-opals.

Obsidian or Moldawite, cut as brilliant, broken in two.

Magnetite : black grain.

Strass : 2 large cherry-red brilliants.

Doubtful : 7 small stones.

THE MINERALOGICAL MAGAZINE
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MINERALOGICAL SOCIETY.

No. 44. DECEMBER, 1891. Vol. IX.

Notes on Crystallites.

By Frank Rutley, F.G.S.
Lecturer on Mineralogy, Royal College of Science, London.

[Read April 14th, 1891.]

THE work of the late Hermann Vogelsang, and the subsequent researches of O. Lehmann and other observers, have done much to w that a careful study of the rudimentary forms of crystallisation may ve as valuable to the mineralogist as that of embryology has already n to the comparative anatomist.

t is, indeed, difficult to form any clear conception of the manner in ch a crystallite originated, nor can we go further back in the history ts development than a stage in which a number of atoms have been uped in an arrangement about which there is yet no certainty, thus ning molecules grouped in some manner about which we are equally ertain. We can but imagine that there is some inherent property in ecules which causes them, under given conditions, to group themselves n orderly manner and to assume a form which may, when more fully eloped, be referred to one of the recognised crystallographic systems.

Vogelsang was the first to indicate that the globulite is the most rud mentary form, revealed as yet by the help of the microscope, from whic crystalline bodies become developed.

The following extract, translated from Rosenbusch's *Mikroskopisch Physiographie*,[1] may serve as an appropriate preface to the suggestion in this paper.

"Vogelsang, who lays stress upon the symmetrical polyhedral form a the principal means for acquiring an understanding of crystals, recognise the development of crystallites and that of crystals as two differen methods of the process of crystallisation, which are united through gradu transition. Whether the one or the other event will take place in a solu tion is, according to his idea, dependent upon the relation between the mol cular resisting-powers in the solution and the molecular movements withi the globulites. The form of the globulite is, however, according to hin the chief condition which determines whether a crystallite or a crystal wi result from it. Between the two an almost infinite number of growth forms occur, interwoven-forms (*gestrickte Formen*), crystal-skeleton ('*Krystalskelette*'), often many-branched, in which the closed form the true crystal is lacking, in which the globulite is no longer recognisab as such, and in which, when symmetrical relations permit it, double refra tion is exhibited.

"In the theoretical discussion, with which Vogelsang accompanies h observations, he does not attempt any special explanation of the formatio of the globulites, and he refers the movements and the arrangement of th developed globulites, as well as their transformation into integral parts a true crystal, to a molecular movement (crystallisation-power) inherent i them, which enables them to attract one another and group themselve regularly."

The globulites, formed in Vogelsang's experiments with sulphur di solved in bisulphide of carbon and Canada balsam, were supersaturate drops of the solution which, on parting with their solvent, became soli bodies, minute, transparent, isotropic pellets of sulphur.

It is customary to regard globulites as balls or spheres, but, when w consider how diminutive they are and how small they appear, even whe magnified over 1,000 diameters, it seems quite probable that a globuli has not in all cases a perfectly spherical form, but may sometimes be a ellipsoid. The difference between a sphere and an ellipsoid of such sma dimensions would barely be appreciable, even under very high magnifyin

[1] 2nd Ed. Vol. I, p. 28.

ers, and it does not, therefore, seem unreasonable to suppose that t we term globulites may include ellipsoids of rotation and triaxial soids as well as spheres.

etrographical literature has already made us familiar with these ute bodies, individually known as globulites (Fig. 1), collectively as garites (Fig. 2), longulites (Fig. 4), &c. In addition to these a vast ber of other microscopic bodies have been described and figured, but o the present time no distinctive names have been given to them. At risk, therefore, of encumbering petrography with new terms I venture uggest some of the names given in the annexed table. It would, ed, be next to impossible for me to explain what follows without re- se to such terms. The names already in use are underlined.

nis table may serve as a provisional but very crude classification of allites, in which four groups or stages are recognised. These may egarded as shading off one into the other, as there are no sharp lines marcation between them.

I. Primitive Stage.

Globulite.

Margarite. Clavalite. *Longulite.* Spiculite.

Bacillite. (*Belonite.*)

II. Spherulitic Stage.

Cumulite.

Spherulite. *Globosphærite.*

Axiolite.

Scopulitic Spherulite. *Trichite.*

III. Setulate Stage.

Scopulite.

IV. Chiasmolitic Stage.

Arculite. Furculite.

½ Arculite. Biconcave Rotulite.

¼ Arculite. Crenulite.

ne first group or stage includes globulites, and those elongated forms h result from the coalescence, partial or complete, of several, some- s of many globulites. These forms are too well known to need any ription. The name clavalite is here given to club-ended longulites 3), or dumb-bell shaped bodies, the term longulite being here icted to cylindrical rods with rounded ends and of uniform dimensions end to end of the cylinder. The bodies to which the name bacillite ven are of much greater size, and apparently consist of a number of ulites massed together with their longest axes parallel (Fig. 6). The

term is convenient as indicating a rod-like body which does not prese any recognisable crystallographic faces. The cumulite (Fig. 29) might placed in this stage, but since these bodies appear at times to pass in globosphærites, they may be regarded as forming a connecting link betwe Stages I and II. They are here placed in the latter.

In Stage II. the cumulite (Fig. 29), or irregular aggregate of globulite may, by accession of fresh globulites, ultimately form a spherical bo known as a globosphærite (Fig. 30). Generally, however, spherulites ha a radiating fibrous structure (Fig. 26), the fibres or minute crystalline ro diverging from a point or from several points situated at or about t centre of the spherulite. The rods may be either straight and single they may split up into delicate branches. The surfaces of spherulites a sometimes smooth, at others rough, while occasionally their irregulari of surface is so great that their sections present roughly-jagged or tooth boundaries (Fig. 27), and in some cases, where the rock is almost whol composed of them, they meet in polygonal boundaries. In these spher lites, with a radiating fibrous structure, we may assume that the separa fibres or rods have been formed by the linear arrangement of globulite and in this respect they differ from those spherulites in which the glob lites do not appear to have any definite arrangement, as in the glob sphærites already mentioned. The elongated bodies known as axiolit possess the radial structure of a spherulite only to a limited extent, a that merely at the extremities of the axiolite (Fig. 25). Throughout t remainder of the body the fibres are radially disposed about a central ax so that they lie at right angles to the roughly cylindrical surface of t axiolite. In general appearance they somewhat resemble spherulit which have been drawn or squeezed out of shape.

If we conceive the equatorial constriction of a plastic spherulite with radiating fibrous structure, we can imagine that, according to the extent the constriction, we should compress the fibres lying at right angles the ligature and thus drive the remaining fibres into a sheaf-like for By using a comparatively broad ligature we should form a rod, the leng of which would be approximately proportional to the breadth of t ligature. The fibres passing from the ends of this rod would diver around the poles of the compressed sphere and would sweep in curv towards the zone of constriction. This structure is not uncommon in so spherulites. Those shown in Figs. 23 and 24, occurring in a spheruli lava from Mokua Weo Weo, in the Sandwich Islands, are good exampl

This kind of spherulite forms a connecting link between Stages II. a III. The trichites (Fig. 28), which have been provisionally placed in t

ue group with the spherulitic forms, may result either from the group- of margarites about a common centre, or from the abnormal growth, as Lehmann has shown, of fresh material upon the terminations of crys-, so that they pass beyond the halo of crystallisation into the super- rated mother liquor or magma, in which the currents bring such a ntiful supply of fresh material that the thin needle or thread increases y rapidly in length, while the impoverished solution streaming down its es affords no material for increasing the breadth of the filament, the nature of which he considers to be due to the action of currents.

Bearing the scopulitic-spherulite (Fig. 23) in mind, let us next turn our ntion to the third or Setulate Stage, in which certain bodies occur ch I have ventured to name scopulites in allusion to their brush-like racter. In structure they appear to differ in no essential respect m the scopulitic spherulites, except that we can see little in them ch suggests any relation to spherulites. This is, however, merely due he length of the rod which carries the two brushes or plumes (Figs. 8 9), and if we wish to compare the very variable separation of the shes, and consequently the variable lengths of the rods which carry m, we have only to examine a section of the well-known pitchstone of iegills, in Arran. Here we may see these scopulites in abundance; e with the brushes separated by long rods, others with the shortest sible stems. They frequently occur fringing the long spicular crystal- and spike-like rods of hornblende with their greenish tufts, and cause n to resemble the fronds of ferns, as in Fig. 10.[1] Scopulitic structure also be seen in the hornblende of certain phonolites, hornblende sts, &c., the prisms splitting up at the ends into divergent fibres, as in 7.

Keeping this last or worn-out paint-brush type of scopulite before us, pass on to the fourth or Chiasmolitic Stage. Here we have two forms very common occurence in vitreous rocks, slags, &c., which I have ned arculites and furculites (Figs. 11 and 14) on account of structural ularities which respectively characterise them.[2] Of the two the arcu- is the more closely allied to the scopulite. The leading characteristic of arculite is the presence of four main ribs, like bows set back to back so their concavities are facing outwards (Fig. 11). This structure may be

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[1] This is a diagrammatic figure. These growths are frequently irregular. They etimes occur as mere arborescent growths, as in Fig. 35, and are then barely gnisable as scopulites.

[2] Forms of this kind are described by Krukenberg as chiasmolites. "Mikro- hia der Glasbasalte von Hawaii." Tübingen, 1877.

seen in crystallites or crystal skeletons of felspar, olivine, &c., met with in certain volcanic rocks, usually vitreous. The structure, however, is not confined to crystallites, for distinct evidence of it may often be seen in well-formed crystals, as in the felspars of some trachytes, in which the skeleton structure, although its interstices are now filled in, is still clearly apparent. The four bow-like main ribs of the arculite are sometimes so close together that their convex sides are in actual contact, at others they are more or or less widely separated.

From their extremities towards the middle of each bow delicate comb like growths are developed, those proceeding from the one bow following the direction of curvature of the adjacent bow. They are, as a rule, but slightly developed towards the middle of each bow. In section two bow or main ribs only appear, but in reality four are present in each fully developed arculite.[1] Sometimes a small crystal form may be seen to occupy a position midway between the bows, and it not unfrequently happens that the bows are small or dwarfed, forming mere horns on the solid angles of the diminutive crystal form.

The arculites are very closely related to the furculites, which I have so named on account of their fork-like structure.[2] In the latter there appears to be a central main stem, which forks at its extremities into straight prongs which, as a rule, enclose tolerably wide angles (Fig. 14). At each extremity of the main stem there are four of these prongs. Two of these only are seen at each end of the main stem in microscopic section, when the plane of section is parallel to the main stem.

Pectinate growths proceed from each prong, those of the one prong apparently following the direction of the adjacent prong. The general structure so much resembles that of the arculite, that there is evidently a close relationship between these two forms.

In a furculite occurring in a section of Mexican obsidian there is a repeated tendency to develop scopulitic structure, the directions of the scopulitic fibres approximating to those of the pectinate growths of the furculite (Fig. 32).

The lath-shaped microlites so frequently seen in microscopic sections of volcanic rocks also seem, at times, to be in some respects related to furculites, especially those which, owing to an excessive growth from the solid angles, have developed projections of various kinds, sometimes

[1] In hemihedral forms, however, only two main ribs would be present.

[2] The forms which I have here termed arculites and furculites were named chiastolites by Krukenberg, and I have in consequence applied the name to the group or stage in which they occur.

ow-tailed in form (Figs. 17 and 20), at others stepped or crenulated .. 18 and 19). I have provisionally included them all in the pre- g tabular classification under the name crenulites.

ied to the furculites and arculites, but especially to the latter, are peculiar forms which at first sight appear to belong to a separate . It would seem, however, that they are hemihedral and tetar- ral forms derived from the arculite.

at shown in Fig. 13 appears to be derived from the arculite by the opment of alternate pairs of octants, while that in Fig. 12 seems ult from the development of two octants (2 and 8, 1 and 7, 3 and 4 and 6), one above, and the opposite one below the diametral

e forms which we have now been considering comprise, I believe, the types upon which crystallites are built up. Doubtless there are s which I have never seen, or which may have been overlooked. e is, however, one peculiar form which I think is worth description. e very remarkable example of discoidal crystallisation is afforded by rtial devitrification of a green glass from the St. Helen's Works, ing in a large series of specimens forwarded to me by Mr. Douglas n for examination. This specimen, which is of a clear deep green , contains a great number of white, biconcave, circular discs, some ich occur singly (Fig. 15a), while others form interpenetrating s, one disc lying at right angles to the other with the margin of the assing through the centre of the other (Fig. 16). At other times nterpenetrate at an angle less than 90°, but as they are more or eeply embedded in the glass, the angles at which they intersect can- adily be measured. These bodies may clearly be seen by unassisted or by the help of a pocket-lens.

large proportion of them appear massed together in a confused r like heaps of diminutive coins. The concave surfaces of duals occasionally meet or intersect, so that a small speck of glass may be seen to occupy the centre of the disc. Where the nen is broken and the fracture passes through one of these bodies, t angles to its circular section, as in Fig. 15b, its biconcave nature rly revealed. For such a body I propose the name biconcave-rotu- Whether these rotulites bear any relation to spherulites has yet to

e curved stems of these forms appear at times to consist of longulites, so d that they overlap like tiles on a roof (Fig. 31). See Explanation of Plate.

hen this paper was read the term biconcave-discolite was employed. Since, er, Prof. Judd objected to the term on the ground that it was already used to ate certain microscopic bodies occurring in calcareous deep-sea deposits, the otulite has been substituted.

be determined, but in structure, at all events, they are quite different. the biconcave-rotulite any section passing through its centre and at rig angles to its circular section shows a structure which, in some cases, precisely that of an arculite, in others that of a furculite. It would see indeed, that this form would result from the revolution of an arculite of a furculite about an axis normal to the main stem.

We may assume that one section of the rotulite is circular, judgi from the appearances which these bodies generally present in the har specimen. In a section cut from it, however, there are some rather sm transparent sections which are decidedly hexagons, and on measuring t angle at which some of the more delicate crystalline growths, lying in t interior of the hexagon, intersect, it was found to be 60°. Such an he gonal section may be derived from the interpenetration of three rotuli of equal dimensions at 60° to one another.

In the forms of the Chiasmolitic Stage which have just been describ the relation of the arculites, furculites, and rotulites appears to be a v close one. If now we compare the various forms of crenulites with th of arculites and furculites, we see in the boundaries of the former a cert approximation to the forms of the latter. Compare, for instance, Fig. with Fig. 11, and Figs. 19, 20, and 21 with 14, and we can picture in crenulites the partial or complete filling in of the skeleton-forms of ar lites and furculites. In connexion with this it may be interesting to n that if a glass solid of the form which would result from the filling in furculite, such as that represented in Fig. 14, be devitrified, the lines which the arrest-planes[1] meet would correspond in direction with the m stem and prongs of the furculite if the rate of the growth in the la were equal in all directions, and a section through such a devitrified pri would resemble that shown in Fig. 34. In this instance, however, have the lines, formed by the meeting of the divisional planes in the so resulting from the development of crystalline rods or bundles of crystall rods, which, originating on the limiting surface of a polyhedral form, p *inwards* until those proceeding from one surface arrest those proceed from another surface, while in the filling in of a furculite we have grow which take place *outwards* from pre-existing lines. The one process in fact, the reverse of the other.

A few of the figures given in the plate have not yet been alluded to, have been inserted as illustrative of certain structures already noti The figures are briefly described in the explanation of the plate.

[1] "Anniversary Address, Geological Society." By Prof. T. G. Bonney, *Proc. Royal Soc.* Vol. 39, p. 90. Herman and Rutley, 1885.

conclusion, the chief points suggested in this paper may be sum-
sed as follows :—

age I.—The Primitive Stage comprises those bodies only which are
r individual globulites, aggregates of globulites, forming elongated,
liform, cylindrical or fusiform bodies, or bacillites resulting from the
lel grouping of such elongated forms. In no case has the globulitic
ent assumed a polyhedral form in this stage.

age II.—The Spherulitic Stage results from the grouping of members
e primitive stage into bodies which are usually more or less spheri-
and sometimes with, at other times without, a radial arrangement.
sionally these bodies show considerable departure from the spherical
, such differences being due to the same causes which determine the
orphic and allotriomorphic characters of crystals.

age III.—The Setulate is closely related to the preceding stage, the
ecting link being the scopulitic spherulite. No pectinate growths
present in the scopulites, and they differ from one another only in
development and curvature of their constituent fibres, and in the
th of rod by which their brushes are separated.

age IV.—The Sagittal Stage is related to the preceding, but less
ly than at first sight appears. The fibres or rods of the scopulites do
ear any apparent relation to the curved main ribs of the arculites, or
e forked main ribs of the furculites, and consequently cannot be
ed as indicating definite arrest-planes. In the forms of the sagittal
, which are essentially skeleton crystallites or crystals, there is a
skeleton which seems to correspond with the boundaries of the
t-planes seen in similar glass solids when devitrified, these arrest-
s bearing a definite relation to the symmetry of the body in which
occur. Pectinate growths proceed from the limbs of the main
tons of sagittal crystallites. No such growths are met with in the
ate stage.

so shortening the central rod of the main skeleton of a furculite that
oints from which the upper and lower series of forks originate are
ght together, the central rod becomes abolished. If now the alternate
of the upper series be brought together and also the alternate forks
lower series, so that they coincide or unite to form a single rod, the
ton will then become that of a pyramid or other closed simple form.
ersely, by separating the skeleton of a pyramid into forks, and by
evelopment of a central rod separating the upper series of forks from
ower series, the skeleton of the corresponding prism is formed.

seems probable, therefore, that the sagittal forms represent the frame-

work of all polyhedral crystallites or skeleton crystals. Much remains be done in working out the relations of the forms.

How far the views expressed in the foregoing very imperfect sketch m be borne out by known facts or by further observations, is a question abo which there is probably some doubt, and it may be that I have speculat far too freely and generalised to an unwarrantable extent upon the limit material at my disposal.

Explanation of Plate.

Typical Forms of Crystallites.

Primitive Stage.

1. Globulite. 2. Margarite. 3. Clavalite.
4. Longulite. 5. Spiculite. 6. Bacillite.

Setulate Stage.

7, 8, 9. Scopulites. 10. Scopulitic growths occurring in Pitchsto Corriegills, Arran.

Chiasmolitic Stage.

11. Arculite. 12. $\frac{1}{4}$ Arculite. 13. $\frac{1}{2}$ Arculite. 14. Furculite. 1 Biconcave-rotulite, St. Helen's Glass-works. 15b. Section of sa showing arculite structure. These bodies sometimes possess the struct of furculites.

16. Biconcave-rotulites interpenetrating apparently at right ang St. Helen's Glass-works. 17, 18, 19, 20. Crenulites.

21. Same form as No. 20, but with the re-entering angles filled in.

22. Lath-shaped microlite.

Spherulitic Stage.

23. Scopulitic Spherulite.

24. Ditto, showing irregular boundary. Lava of Mokua Weo W Sandwich Islands.

25. Axiolite, 26. Spherulite. 27. Spherulite with jagged bound Felsitic Lava, Glyder-Fawr, North Wales. 28. Trichite. 29. Cumulite. Globosphærite.

Supplementary Illustrations.

31. Portion of slender undulate rod, the concave aspects of which s an imbricate arrangement of longulites. The orientation of the latte the same on either side of the central or nodal point, but in the surrou ing glass detached or barely adherent longulites may be seen; those one side of the rod differing in their orientation from those on the o side. On focussing $\frac{1}{3500}$ inch lower, two other fringes of barely adhe longulites are brought into view, but they are orientated in directions reverse of those seen at the higher focus. These fringes extend along

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Rutley del

ved rod for about $\frac{1}{3}$th of its entire length. Pitchstone, Corriegills, an. × 780 linear.

2. Portion of crystallite showing furculitic structure (F); passing into ulitic structures (S). Obsidian, Mexico. × between 150 and 200 neters.

3. Crystal of sanidine, twinned on Carlsbad type and showing divi- al markings suggestive of furculitic structure. Trachyte, Berkum, Bonn, Rhine. × cir. 80 diameters.

4. Diagrammatic vertical section of a square prism of glass, com- ly devitrified and showing the traces of arrest-planes joined by a cen- line of arrest which coincides with the principal axis of the prism. resemblance of the divisional lines in this section to those observed in tallites and crystals which show a furculitic structure is at once seen omparing this figure with Figs. 32 and 33.

5. Irregular arborescent growths upon portion of a slender rod of horn- e in pitchstone, Corriegills, Arran. Examples are seen in this rock hich the direction of curvature of these growths upon one side of the s the reverse of that seen upon the opposite side. Such growths be regarded as imperfectly developed scopulites, which, growing at ntervals and springing obliquely from their points of attachment, are le to develop the perfect scopulitic form.

, 37, 38, 39. Skeleton crystals in basalt-glass, Kilauea, Sandwich ds. Three different sections, apparently of the same crystal form, are n in Figs. 37, 38, and 39. Fig. 39 is probably a section approxi- ly at right angles to the vertical axis.

. Section similar to Fig. 39 (embryonic form). Basalt-glass, Kilauea. 50 linear.

. Skeleton crystal, Basalt-glass, Kilauea. × cir. 250 linear. Such a might apparently be derived from that shown in Fig. 37 by shorten- e central rod of the latter.

. Skeleton crystal, basalt-glass, Kilauea. × cir. 250 linear, with closed crystal-form developed in the centre.

. Portion of crystal of sucrate of lime, showing arculite structure. direction of maximum extinction makes an angle of 23° with the ver- axis.

Devitrification of cracked and brecciated Obsidian.

By Grenville A. J. Cole, F.G.S.
(Professor of Geology in the Royal College of Science, Dublin.)

[Read April 14th, 1891.]

In examining a specimen of the rhyolitic lava-flow of the Rocche Ross Lipari,[1] in the collection of the Royal College of Science, London, was struck by the fact that the areas of pure black glass visible upon th fractured surface were each bordered by a band of spherulitic matter, th convex surfaces of the spherulites being directed inwards towards th glassy area. The specimen, collected by Prof. J. W. Judd, F.R.S evidently comes from a portion of the flow which has become brecciate during movement of the viscid mass; intermingling of glassy and hem crystalline matter has occurred, and an approach to the structure of th well-known "Piperno" of Naples has resulted. But the irregular litt lumps of obsidian have not received their spherulitic crust from an chilling effect that they may have had upon the surrounding groun mass; the spherulites belong to the glass-fragments themselves, and are n deposited upon their surfaces.

So much can be seen in the hand-specimen. Prof. Judd has kind had a thin section prepared, which shows that this structure of glas fragments with hemicrystalline envelopes is also a microscopic feature.

Thus the large lumps of obsidian exhibit a thin zone of spherulit matter following every curve of their outline; the particular sectio examined has not traversed any examples of the thicker crusts so easi visible to the unaided eye. But the hemicrystalline interstitial matte playing the part of groundmass in the specimen, unexpectedly exhibi the same characters on a minutely detailed scale. One or two lumps dull normal rhyolite occur; but the main mass appears at the first sig to consist of well-developed axiolites, fitting closely against one anothe irregular and curving, as in the fine examples from the Western States America.

[1] See Prof. Judd, *Volcanoes*, p. 124.

axiolites, however, spherulitic aggregation has taken place along an
the body grows from this line outwards, and convexities of the sur-
occur where the normal spherulitic rather than the axiolitic tendency
ts itself. In the rock of the Rocche Rosse we soon see that the
daries of the separate bodies are cracks, or even accidental lines of
ion of adjacent fragments, and that the fibrous crystallisation has
nated at the surface and spread gradually inwards, often leaving a
y area at the centre. The minuter structure of the rock is thus merely
roduction of that of its coarser portions.

e spherulitic envelopes, thus found to be so prevalent, are subse-
in origin to the fracturing and partial brecciation of the mass. A
axitic" structure, like that of the Piperno or of the pitchstone of
tshausen in Saxony, having been set up in this layer of the lava-
sufficient heat remained in the mass to promote slow devitrification.
surfaces of the displaced fragments of obsidian, and the cracks which
in the deformed and ruptured bands of glass, have served as starting-
for the crystalline tufts and spherules, which have only very rarely
d into the centre of the glassy particle. In this matter we have
trong support of the experiments of Messrs. Herman and Rutley[1]
e devitrification of artificial glass. The glass operated on, of
ver form, becomes opaque and crystalline from its surface inwards
exposed to prolonged heat short of the temperature of fusion.
fibrous groups of crystallites also develop along cracks;[2] and a
y microscopic fracture suffices for their production, as was shown by
xperiment of flawing a piece of plate-glass beneath the scratch of a
nd, and subjecting the specimen to a bright red heat for nine days.[3]
e, in the case of the lava of the Rocche Rosse, subsequent heating
have even fused the surfaces of adjacent fragments together; but the
hat a thin crack separated them before devitrification set in would
nt for the present disposition of the products of that process.

. Rutley's experiments upon natural glassy rocks[4] confirm us in our
on that spherulitic crusts will arise where fragments of obsidian,
ently consolidated for fracture, are subjected to renewed heating, or

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[1] "On the microscopic characters of some specimens of Devitrified Glass." *Proc.*
oc. Vol. XXXIX. pp. 87-107.

[2] *Id.* p. 99.

[3] *Id.* p. 107.

[4] "Notes on alteration induced by Heat in certain Vitreous Rocks." *Proc. Roy.*
ol. XL. p. 434.

even to the slow passage of the residual heat from the lava-flow of wh they form a part.

The structure above described merits consideration when cases axiolites are under examination. On a microscopic scale it is possible mistake the crack or line of junction between two obsidian fragments the axis of an axiolite, and the two completely independent bands spherulitic matter on opposite sides of it for a united axiolitic body.

Lastly, perlitic structure seems to arise in glassy lavas after the sphe litic stage has been passed through, or where it has been rendered i possible by the rapidity of cooling; but a perlitic glass subjected to newed and sufficient heating ought to present very interesting and strik features of the Rocche Rosse type.

Section of Obsidian of Rocche Rosse, Lipari.
× 18.

In the upper part of the field is a large glassy fragment, on the margin of wh a thin spherulitic envelope has developed. The main part of the field is occu by smaller fragments of glass, which have also become devitrified from their o surfaces inwards.

he Occurrence of an Aluminous Serpentine (Pseudophyte) with flint-like appearance near Kynance Cove.

By Howard Fox, F.G.S.

[Communicated by Mr. Teall. Read April 14th, 1891.]

YNANCE Cove lies in the midst of a Serpentine District. The Serpentine is associated with some little hornblende schist in the cove, is traversed by black dykes both porphyritic and non-porphyritic a few inches to several feet thick, by granitic veins, as also by veins ussuritic gabbro, which latter, are, however, of rare occurrence on the ern side of the Lizard.

hen looking due east from the middle of Kynance Sands a rectangular of white rock is seen in the dark red serpentine cliff from 70 to 80 bove the sea-level at low water, and about 200 yds. from the spectator. vein that gives rise to this block can be easily reached by crossing the m that enters the cove from the east, climbing the rising ground, and reaching a ridge about 100 yds. from the stream, descending the y slope to the right to a point about 200 yds. south-east of Oliver's eshment House. Here, a few feet below the grass, a block of rock is 4 ft. high, 6 ft. through in front, and covered by broken serpentine nd. This block, which we will call No. 1, weathers in places an ue white, like felspar, and in other places like broken flint. On uring it, some of the specimens obtained are found to be composed compact white substance, whilst others have the appearance of typical from which they are easily distinguished by their softness. The ue white rock is the hardest, and the flint-like rock the softest. This can be traced for 40 yds. running down the cliff in a south-south-direction at an angle of about 13°. The serpentine has jointed out the run of this vein, forming somewhat of a cleft, which renders the nination of the vein an easy matter. Twenty-five feet south-south- of this block "No. 1" some more of the same rock is seen embedded in ight green soft decomposed serpentinous material which is so often ciated with any disturbance in the serpentine of this district. The

vein then becomes nipped out for a few feet, to reappear in the form blocks of flint-like rock varying in size downwards from 2½ ft. long by 1 f thick, embedded in decomposed material as before. Sixty-four feet soutl south-east of block " No. 1 " we find the the large block " No. 2," whic is so conspicuous an object from the sands. This block is 9 ft. long b 4 ft. thick, and lying behind it and connected with it is another bloc 2 ft. thick, which would indicate an original vein 6 ft. in thickness. Th western face of this No. 2 block weathers like felspar, the rest weathers dark colour. Some of the flint-like portions are translucent and of greenish colour. These portions are much jointed, and in consequence this the surfaces are roughened by angular projections. Some of th pieces broken from this block have the lighter and darker rocks in con centric layers, whilst others have alternate bands. A knife mak no impression on some of the opaque white rock, whilst some the translucent rock yields to the nail. Fifteen feet beyond this " No. 2 block, the vein decreases to 1 ft. in width, and stones of flint-like rock a embedded in soft green material. Seven feet further on the vein fork and encloses a block of serpentine, the entire width of both veins and th included serpentine being about 2ft. Six feet further on the vein agai branches, enclosing in its course masses of serpentine from 2½ to 3 f thick ; the lower branch of the vein is from 4 to 6 ins. thick, showin much decomposed rock surrounding the flint-like rock ; the upper branc is 6 to 9 ins. thick, and contains much flint-like rock in serpentinou material. The branches of the vein can be traced from this point a small recess or cave into the back of which the upper vein is seen disappear, having a dip of about 35° east-north-east. The end of th vein is about 120 ft. south-south-east of block No. 1.

About 20 ft. below this cave a vein of very coarse-grained saussurit gabbro may be traced for several yards running parallel to the flint-lik vein. This gabbro vein is 15 ins. thick in one place, and thins away nothing. The edges of the saussurite in this vein become in some place beautifully translucent and of a light green colour, and bear a resemblanc to the translucent angular projections of the flint-like rock of bloc " No. 2," described above. In fact there is a passage of the saussuri into the flint-like substance which forms the subject of this paper. Th gabbro has, however, no connection with the flint-like vein. Ten fe south west of this gabbro vein is a black dyke from 3 to 9 ins. thic running in a parallel direction. Further to the south, in the roof a high cavern, an inaccessible oval block about 20 ft. long weathers in th same manner as the flint-like rock. Other black dykes occur here also.

There is one other exposure of the flint-like rock. In ascending the ng ground after crossing the stream that enters Kynance Cove from the t (already referred to), we find, a little more than half-way up, say ut 50 yds. above the stream, that the foot-path through the grass has bare some feet of this flint-like rock. One specimen is very soft, is scratched by the nail. It is associated with and apparently passes opaque white rock which is not scratched by the knife. This expo- is in the trend of the flint-like vein, and about 100 yds. north-north- of block No. 1.

Ir. Player kindly analysed the flint-like rock for Mr. Teall, who has ished me with the following result and analyses of some pseudophytes ch supply an interesting comparison with that of this rock.

		Pseudophyte.		
	I.	II.	III.	IV.
SiO_2	33·3	33·51	34·63	35·31
Al_2O_3	21·8	15·42	17·13	18·28
Fe_2O_3	·4	—	—	1·26
FeO	—	2·58	1·61	·83
MgO	29·7	34·41	33·38	31·61
H_2O	14·9	13·21	13·93	13·26
	100·1	99·13	100·68	100·55
Sp. Gr.	2·57 2·54	2·61	2·58	2·61

I. Flint-like rock from Kynance.

II. Pseudophyte from Zdjar Mountain, near Aloysthal, Moravia.[1] uer)

II. Greyish-green substance from Plaben, near Budweis.[2] (Drasche)

V. Similar substance from Chyn, Bohemia.[3] (Gintl)

Dr. A. Kenngott, *Mineralogische Notizen Sitz. d. k. Akademie d. Wissenschaften.* n. Band XVII. (1855), p. 170.

Dr. R. V. Drasche, Ueber eine pseudomorphe Bildung nach Feldspath. *Mineralo- ue Mittheilungen; Wien* 1873, p. 125.

Quoted from Rammelsberg *Handbuch der Mineral-Chemie.* 2nd edit. Leipzig, .

The Optical Indicatrix and the Transmission of Light in Crystals.

By L. Fletcher, M.A., F.R.S.,

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Late Fellow of University College, Oxford.

[Read June 16th, 1891.]

Section I.—Recent Change of View as to the Properties to be assigned to an elastic luminiferous Ether.

Deduction of the form of the wave-surface for biaxal crystals.

Fresnel's representation of the laws of transmission of rays of light in biaxal crystals, by reference to the surface distinguished by his name, has long been regarded as one of the greatest achievements in the domain of Physical Science. In his memoir[1] on Double Refraction, Fresnel proceeded as follows:—

1. He assumed that the transmission of a ray of light is effected by means of an elastic ether vibrating transversely to its direction.

To the ether is thus assigned a property not belonging to a perfect fluid body in a state of rest: perfect fluidity of a body at rest involves incapacity of resistance to mere change of shape, and it is to such distortional resistance that transverse vibrations must be due.

2. He assumed that the ether of a crystal, when undisturbed, is a system of equal particles, in stable equilibrium under their mutual attraction, and that, for each pair of particles, the latter depend solely on some function of the distance between them and act in the line joining the centres.

He showed that in a medium so constituted there are at least three directions, at right angles to each other, such that the force necessary for the maintenance of a small displacement of a single particle of the ether along any one of them will act in the line of the displacement and be proportional to it in magnitude: that the elastic force evoked by the displacement of a single particle of the ether through unit distance along each of these directions may be different, a^2, b^2, c^2, respectively: that in this case, which is assumed to be that of

[1] *Mémoires de l'Acad. de l'Institut de France*, 1827, vol. 7, pp. 45-176.

in a biaxal crystal, the elastic force due to the displacement of a particle of ether in any direction distinct from the three already oned will act in a direction different from that of the displacement; If the direction of a radius vector of the surface $a^2x^2 + b^2y^2 + c^2z^2 = (y^2 + z^2)^2$ represent that of the displacement of an ethereal particle, he corresponding elastic force for a displacement through unit dis- be resolved along and perpendicular to the line of displacement, rmer component is proportional to the square of the radius vector gnitude: that for displacements of a single particle in directions in a given plane passing through the centre of the above surface, lastic force is generally obliquely inclined to the plane, but that are always two directions, namely those of the longest and shortest ters of the section of the above surface by the given plane, for which solved component of the elastic force in the given plane acts in the f displacement.

He assumed that the ether is virtually incompressible for the forces rned in the transmission of light.

lecting, therefore, the component of the elastic force normal to a containing a set of similarly displaced particles (wave-front) as being t effect by reason of the incompressibility of the ether, Fresnel in- that, for particles in the given plane, vibrations parallel to either ngest or shortest diameter of the corresponding section of the above e must be persistent, since the only effective component of the elastic for each particle then acts in the direction of the displacement.

From a suggested but forced analogy of a line of vibrating ether- les to a vibrating string, Fresnel assumed that the velocity of trans- e of a wave-front along its normal is directly proportional to the of that principal diameter of the section of the above surface by ave-front which is parallel to the direction of vibration.

nce finding, by the usual mathematical process, the envelope of planes enting the positions to which wave-fronts, with every possible direc- would arrive after the lapse of the same interval of time, Fresnel ded that the wave-surface for a biaxal crystal is represented by the on

$$\frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2} = 0:$$

r, as the front corresponding to any ray is parallel to the tangent to the wave-surface at the point where the ray meets it, the vibra- , in general, obliquely not perpendicularly transverse to the direc- f the ray.

5. Hence Fresnel also inferred that the velocities of the two rays whi can be transmitted along a given direction are directly proportional to t axes of the ellipse in which the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is intersected a plane normal to the common direction of the rays.

Its singularities of form.

The closed surface represented by the above equation is of very peculi form, and consists of two concentric ellipsoid-like sheets, which are symm trical with respect to three rectangular planes. There are four points co mon to both sheets; they are situated at the extremities of two diamete lying in one of the planes of symmetry: in the neighbourhood of ea of these points the sheets are drawn towards each other, and the surfa has there the shape of a double cone; an infinite number of tange planes to the surface can thus be drawn at each of them. Furth two planes and their parallels respectively touch the surface, not at o point nor at two points, but at an infinite number of points which lie the circumference of a circle.

These geometrical singularities of the wave-surface, first noticed by S William Hamilton five years after the death of Fresnel, point to t existence in biaxal crystals of certain optical characters which had to that time remained undiscovered, and seemed too strange to be rea the establishment of their actuality by Lloyd has been regarded as t crowning triumph of Fresnel's theory of double refraction; for not on are the phenomena strange, but their observation demands a combinati of circumstances which places them beyond the range of accidental d covery.

Dynamical difficulties of Fresnel's theory of double refraction.

As continued experiment and precise observation have served only establish the high degree of accuracy of the form assigned to the wa surface by Fresnel,[1] it might naturally be inferred that the assumptio which lead, after so elaborate a course of reasoning, to a surface presenti these singularities must be themselves beyond cavil. Yet, strange to s the mathematical process, by which the surface is thus arrived at, is of which the weakness was recognised by the author himself, and

[1] Kohlrausch: *Wied. Ann.;* 1879, vol. 6, p. 86; vol. 7, p. 427.
Glazebrook: *Phil. Trans.;* 1879, vol. 170, part 1, p. 287. *Proc. Roy. So* 1883, vol. 34, p. 393.

ory has long been regarded as dynamically unsound; further, the racters assumed for the ether, though they lead to the true wave-face,· have since been found to have for necessary consequences other eal laws which are inconsistent with the results of experiment. On other hand, the same form of wave-surface can be arrived at from other of assumptions, which have thus the same claim to recognition;[1] yet are inconsistent with those of Fresnel, and with each other. As the hypotheses which lead to Fresnel's wave-surface have been found to other consequences which are contradicted by experimental results, comparative simplicity and the historical interest of the method Fresnel have sufficed to secure the adoption of his assumptions and esponding terminology in the general literature relating to the optical racters of crystals.

he fact that Fresnel's wave-surface has been deduced from several nsistent sets of assumptions as to the characters of the ethereal on suggests that the form may really depend on the feature common l, namely, the transmission of a periodic change of state differently ed to different sides of the ray, and is otherwise independent of the ical character of the transmitted change: the suggestion is discussed ection V.

Results of the rigorous calculation of the vibratory motion of an elastic solid.

he rigorous calculation[2] of the vibratory motion of the parts of an iso-c elastic solid is found to involve two quantities, which are generally ted by A and B: the latter, B, measures the rigidity, or the resistance e body to simple change of shape, or the *elasticity of figure;* the er, A, is connected with B, and with k (which measures the resistance mple change of volume, or the *elasticity of volume*), by the relation $-\frac{4}{3}B$. Further, it can be shown that a vibratory motion of the s of an elastic medium generally gives rise to two kinds of waves, respectively to distortional and condensational-rarefactional vibra-; the former travelling with velocity $\sqrt{\frac{B}{\rho}}$, the latter (which correspond ose of sound) with velocity $\sqrt{\frac{A}{\rho}}$, where ρ is the density of the medium. , if the transmission of light through a singly refractive medium be

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[1] *e.g.* Challis in the *Trans. Camb. Phil. Soc.;* 1847, vol. 8, p. 524.

[2] most Valuable Report by Glazebrook on Optical Theories is published in the *Brit. Assoc.* for 1885, pp. 157-261.

due to the vibratory motion of an isotropic elastic solid, all the energ persists in the form of distortional vibrations perpendicular to the ray hence the characters of the ether must be so assumed as to secure th absence of the condensational-rarefactional vibrations. For this purpo we may make either of two assumptions, namely, that A is virtually ze or that A is virtually infinite as compared with B: in the former ca the condensational-rarefactional wave is got rid of by making i velocity zero; in the latter case by making the velocity infinite. But was long believed that the former assumption was otherwise inadmissibl for it was supposed by Green and later mathematicians that the quanti $A-\frac{4}{3}B$ is necessarily positive, if the equilibrium of the parts of elastic body is stable; and this is impossible if A is zero, for B essentially a positive quantity: hence it only remained to assume A a therefore also k infinite, and thus the ether to be virtually incompressib

Double refraction could then be consistently explained by a variation the rigidity of the ether of a bi-refractive crystal with the direction; b it was necessary, for dynamical reasons, to assume the vibrations to be not perpendicular to, the plane of polarisation.

On the other hand, Lord Rayleigh[1] has proved that the phenomena d to the scattering of light by small particles require the vibrations of t ether to be perpendicular to the plane of polarisation; he has furth shown that no theory based on varying rigidity can possibly be satisfacto and that the variation of density in different directions in a biaxal cry tal would lead dynamically to a form of wave-surface different from th of Fresnel, if the ether be incompressible for the forces involved in t propagation of the vibrations.

Sir William Thomson's version of the elastic theory.

From this position of dead-lock, according to which the ether must both compressible and incompressible, the theory that the transmission light is effected by the vibrations of an elastic medium has only recen been extricated. At the end of 1888 Sir William Thomson,[2] re-examin the problem of the stability of the equilibrium, found that the condit that $A-\frac{4}{3}B$ is a positive quantity becomes unnecessary, "provided either suppose the medium to extend all through boundless space, or g it a fixed containing vessel as its boundary:" with either of these pro sions, the stability only requires that A should not be negative, and i therefore possible to get rid of the condensational-rarefactional wave

[1] *Lond. Edin. and Dub. Philos. Magaz.*, 1871, ser. 4, vol. 41, p. 451.

[2] *Ibid.*, 1888, ser. 5, vol. 26, p. 414.

ssumption hitherto deemed inadmissible, namely, that A is zero; this ves the compressibility of the ether for the forces concerned in the gation of light. As a mechanical illustration, Sir William points hat "homogeneous air-less foam held from collapse by adhesion to a ining vessel, which may be infinitely distant all round, exactly ful- he condition of zero-velocity for the condensational-rarefactional ; while it has a definite rigidity and elasticity of form, and a de- velocity of distortional wave, which can easily be calculated with a pproximation to absolute accuracy."

arting with the new assumption, Sir William Thomson was able to e correct expressions for the intensities of ordinarily reflected or re- d light: and Glazebrook[1] has since shown that the elastic theory new form fully accounts for dispersion, including anomalous dis- n (like that of cyanin), double refraction, and metallic reflection, and r that it leads to a correct expression for the velocity of light in a g medium. According to the new version, the vibrations of the ether rpendicular to the plane of polarisation, even in biaxal crystals, and always perpendicularly transverse to the ray: further, the matter- les and ether-particles are supposed to react on each other: and if vibrations are synchronous, the former may even be set in appreciable n by the latter. As the reaction of the matter and ether may produce me effect on the motion of the ether-particles as would result from ple variation of the rigidity or density of the ethereal medium, it be- convenient to distinguish between the *actual* and *effective* values rigidity and density.

s clear that the new version of the properties of the elastic ether, er really true or not,[2] is far more satisfactory than any hitherto sug- , and must replace the older versions until a better one is proposed. it becomes necessary, for those who adopt an elastic ether as the of the undulatory theory, to regard (1) the ether as compressible, for the forces concerned in the propagation of light; (2) the actual y and rigidity of the ether as identical for all bodies; (3) the effective y as invariable; (4) the effective density as different in different , and, in the case of doubly refractive crystals, in different directions the same body.

el's line of reasoning, and the terms based upon it, must be abandoned.

the great majority of mineralogical students, the chief value of the

[1] *Ibid.*, 1888, ser. 5, vol. 26, p. 521.

[2] *Ibid.*, p. 538; 1889, vol. 27, pp. 246, 253: *Nature*, 1889, vol. 40, p. 32.

hypothesis of an elastic ether is in the correlation of the phenom observed when light is transmitted through crystals; for which purpos is very desirable that the student should be able to reach the wave-surfa if practicable, by means of elementary reasoning based on observed fa of a simple character. The rigorous calculation of the motions vibrating elastic medium is not a simple process: it involves, ind mathematics of so high an order that the derivation of the wave-surfac this way will always be unintelligible to the ordinary student of cryst On the other hand, the only comparatively simple mode of derivatio the wave-surface, as yet invented, that of Fresnel, depends upon assu tions of incompressibility and varying elasticity which are now deer untrue; and further, involves for biaxal crystals a general obliquit transverse vibration, not in accordance with the latest version o elastic theory. Under present circumstances, the process of Fresnel, e if adopted on account of its great historical interest, must be puzz to the student, and inevitably lead to the acquisition of wrong views as the properties to be assigned to the luminiferous ether; hence it beco necessary to abandon the whole process, and all those terms now in c mon use (ellipsoid of optic elasticity, axes of optic elasticity, coefficient optic elasticity) which are based upon it.

The form of the wave-surface for biaxal crystals was really discovered another way.

The great difficulty in the correlation of the phenomena of the transmiss of light through biaxal crystals, as already stated, lies in the derivatio the wave-surface. The form of the surface is too extraordinary to directly assumed either as a probable one *a priori*, or as suggested by perimental results. If it can be shown that the form of the wave-surface biaxal crystals is suggested by a simple generalisation, independently of particular version of the undulatory theory, and might have been brou in this way within the province of experimental investigation, the gre part of the present educational difficulty will be removed from the path of student. In fact, we shall find that it was really by a process of gener sation, though not indicated in the composite memoir of 1827, that Fres himself was first led to the true form of the wave-surface for biaxal cryst The properties of an incompressible elastic ether were mathematic developed by him after the discovery of the true form of the wave-surf had been made.

Section II.—Evolution of the Optical Indicatrix.

the present Section it is sought to show that a certain surface, here l the Optical Indicatrix, naturally suggests itself as a means of ition of the laws of transmission of light in uniaxal crystals; a generalisation then suggests the possible existence of biaxal crystals, e general nature of their optical properties. The reasoning may be ed as follows:—

General nature of light.

ht travels with finite velocity.

ush of light transmitted from one body to another may thus for a wholly in the intervening space; hence the transmission of light e one either of matter or of change of state of matter.

Light is due to the change of state of matter.

rays of light of the same colour, travelling in the same direction he same line, may annihilate each other.

e the transmission of light cannot be one of matter; it must be a ission of change of state of matter, and the change must be capable esentation by positive and negative quantities.

An ether is necessary.

ht travels across interplanetary space.

ce interplanetary space must be filled with one or more kinds of , capable of transmitting a particular kind of change of state with an us but finite velocity (186,000 miles a second), and for distances ting to millions of millions of miles. We may conveniently assume e extraordinary matter is wholly of one kind, and designate it by a name, *ether*; it must be extremely subtle, for it offers no appreciable nce to the motion of the planets.

Permeation of ordinary matter by the ether.

t is transmitted, but with different velocities, through ordinary

Hence either ordinary matter is itself capable of transmitting particular kind of change of state, or it is permeated by an ether capa of so doing. Having regard to the enormous velocity with which ligh propagated through interplanetary ether and different kinds of ordin matter, we may assume that the same kind of ether is concerned in transmission, and that the variation of velocity and other characters is to the influence of the ordinary matter on the properties of the permea ether.

The change of state is periodic.

If two rays, continually transmitted along the same line, annihilate e other, annihilation again takes place if either ray is transferred thro any multiple of a certain measurable distance along the direction of tra mission.

Hence, so long as a single ray of light is being transmitted along a l the state of the ether at a given instant is the same at all points dist from each other by a certain measurable quantity, which we may den by λ. But the continual uniform transmission of the change of s along the line involves a continual and periodic change of state at e point of the line; the duration of the period being the same at all poi and always equal to the time necessary for the transmission of the cha through the distance λ along the line: if v be the distance of transmis during the unit of time, the period will thus be $\frac{\lambda}{v}$. During a single per the ether at any point in the line of transmission experiences all th changes which belong at a given instant to all points in a length λ of line of transmission.

The characters of an undulation.

Whatever be its physical nature, a periodic change of character at point is termed a *vibration* of the character: its maximum value, *amplitude* of the vibration: the interval of time required for a comp vibration, its *period*: the state at a given instant, the *phase* of the vibrati the relation between the phase and the time, the *law* of the vibrat If, further, the change is being transmitted along a line or *ray*, the figuration of the states at all points of the ray at a given instant is ter an *undulation*: the least part of an undulation which includes all varie of phase is termed a *wave*, and the distance occupied by a wave, a *w length*.

Light is an undulatory phenomenon.

It follows from the above that, in this general sense, light is undoubt an undulatory phenomenon of some kind or other.

Sound is also an undulatory phenomenon.

similar reasoning, it follows that sound is an undulatory phenomenon. riment shows that the transmission of sound is effected by ordinary r, and that the change of character is one of oscillation of the material les, the oscillation being generally solely in the direction of the nission. The properties at any point of a line of transmission of tinued uniform sound, namely intensity, note and *timbre*, must de- on the characters of the vibration at the point, and thus on the tude, period and law: experiment proves that the intensity of a e sound depends solely on the amplitude, and the note solely on the l.

nsity of light depends on the amplitude, colour on the period of the vibration.

ilarly, the corresponding properties at any point of a ray of ordinary intensity and colour, may be assumed to depend on the characters vibration at the point, and thus on the amplitude, period and law: ay tentatively assume, from analogy with sound, that the intensity imple ray depends solely on the amplitude, the colour solely on the l.

Polarisation of light: plane of polarisation: transverse plane.

ordinary light is capable of a change to which there is no parallel e case of sound. A ray of ordinary light transmitted through air es distinctive characters by reflection at a certain angle of incidence a sheet of glass: as tested by reflection at the same angle of inci- from a second plate of glass, it has different properties on different its properties being symmetrical, however, at every point of the o the same two perpendicular planes intersecting in the ray: one planes of symmetry is the plane of incidence and reflection from st plate. As the planes of symmetry of the ray are dissimilar and e experimentally distinguished from each other, that which coincides the plane of incidence and reflection may conveniently be termed *lane of polarisation;* the second plane of symmetry may be dis- shed as the *transverse plane.* A ray having the same characters, er induced, is said to be plane-polarised.

ace the periodic change of the ether at any point of an aerially itted plane-polarised ray of light is not solely related to the on of transmission, and thus differs in kind from that which charac- sound. For the suggestion of the laws of double refraction, er knowledge of the character of the change is unnecessary.

Transmission of plane-polarised rays in glass and analogous media.

If a plate of ordinary glass or any analogous medium be placed with i faces perpendicular to an aerially transmitted plane-polarised ray, the lig which emerges from the glass is found to be still plane-polarised, and t position of the plane of polarisation is found to be unaltered whatever t thickness of the plate: this is still true, if the plate be turned throu any angle round its own normal.

As the direction of the ray within the plate is coincident with the dire tion of the ray before incidence and after emergence, we may thus reaso ably assume that, at all points of the line of transmission *within the pl itself*, the periodic change of the ether is symmetrical to the same t planes; in which case the position of the symmetral planes of the period change is wholly independent of the glass and depends only on the dire tion of the plane of polarisation of the incident ray. A plane-polaris ray transmissible in any direction within such a medium may have a azimuth of plane of polarisation whatever.

Geometrical representation of the characters of a ray of plane-polarised lig

In representing the transmission of a ray of plane-polarised light, o

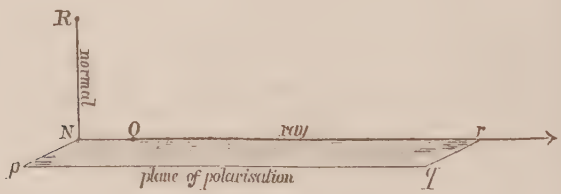

FIG. 1.

single given colour and given intensity, within a given medium, we h thus three characters to consider:—

1. The line of transmission of the ray,
2. The direction of the plane of polarisation,
3. The velocity of transmission.

The direction of a plane being most conveniently defined by the direct of its normal, the above three characters may be geometrically rep sented by means of two intersecting perpendicular lines, one of th definite in position, the other only in direction: and any definite funct of the length of either may represent the velocity.

The direction of transmission $O\,r$, and the plane of polarisation $O\,p$

given ray (Fig. 1), may thus be represented by two lines Or RN, RN is any line perpendicular to the plane $Opqr$, and therefore to the ray Or: the velocity of transmission may be represented by unction of either of the lines Or RN.

O, a point on the ray, be given, and the normal of the plane of sation be taken to intersect the ray, all the characters may be ented by means of a *single line RN, not passing through the given* for the line Or is then known, since it passes through O and ects RN perpendicularly.

The laws of ordinary reflection and refraction accounted for by an undulatory theory.

hundred years ago (1678-90), Huygens showed, by reasoning which lly independent of the physical nature of the periodic change, though gined it to be identical in character with that involved in the trans- n of sound, that the laws of ordinary reflection and refraction of are compatible with an undulatory theory. He assumed that a disturbance of the ether at any given point must eventually produce ances at all other points of the medium, and that in a transparent showing ordinary refraction the velocity of transmission of the dis- ce is independent of the direction; all points on a spherical surface the given point for centre are thus at any moment in a similar disturbance. If we have regard to the arrival of the disturbance its origin, we may say that in this case the *front* of the disturbance epoch is a sphere. The front of the disturbance due to a single may, for the sake of brevity and generality, be called the *wave*. If the disturbance at the centre be persistent and periodic, the which defines the front of the disturbance at a given epoch passes h points of the medium at which, notwithstanding the continual at each point, there is persistent identity of phase of vibration.

gens gave a geometrical construction for the determination of the on of the refracted ray by means of the spherical wave-surface, the n being that of a line joining the point of incidence of the ray to nt of contact of a tangent plane of the wave-surface, drawn through sponding line which lies in the refracting surface and is normal to ane of incidence: if v be the velocity in the first medium, i the of incidence of the ray, and the size of the sphere correspond to se of a unit of time, the distance of the corresponding line from nt of incidence is $\frac{v}{\sin i}$.

The laws of refraction of light by a crystal of calcite accounted for by undulatory theory.

In the case of calcite, the refraction is in general not single but dou One of the rays, and only one, follows the laws of ordinary refraction all directions: hence Huygens inferred that the surface of disturba corresponding to this ray is the same as for ordinary media, namel sphere. It was necessary to assume a different form of surface of turbance to account for the extraordinary refraction of the other ray, the surface which first suggests itself, after a sphere, is an ellipso further, since the refraction of the second ray is the same for all directi equally inclined to a special direction in the calcite-crystal, or since r lying in a plane perpendicular to this line obey the laws of ordin refraction, it is necessary for the surface of disturbance to be one revolution about that direction as axis. Testing this hypothesis and find it satisfactory, Huygens inferred from his observations that the surfac disturbance corresponding to the second ray is really a spheroid, touch at the extremities of its axis the spherical surface of disturbance cor ponding to the first ray. This relation between the surfaces has b confirmed by later experimenters and found to hold for other crys analogous to those of calcite: it is undoubtedly a Law of Nature.[1]

The direction of the extraordinarily refracted ray is given by the s geometrical construction as before, the surface of disturbance being ta as a spheroid instead of a sphere.

The wave-surface is identical with the ray-surface.

Let rs RS (Fig. 2) be two wave-surfaces due to an origin O, and v the line OrR as axis describe a cone of small angle, determining areas AB upon them. There is great difficulty in imagining the exact natur the physical process by which an isolated *ray* could be propagated thro the ether by means of undulations: still the conception of a ray of l comes so naturally, and has been found so serviceable from the v earliest times, that rays, rather than waves, will be used throughout present paper.

Having regard to the apparent rectilinearity of propagation with homogeneous medium, we may reasonably assume that, if light is pr gated by the disturbances of a medium and the disturbances at all part

[1] Stokes: *Proc. Roy. Soc.*, 1872, vol. 20, p. 443; *Comp. Rend.*, 1873, vol. 77, p. Abria: *Ibid.*, p. 814. Glazebrook: *Phil. Trans.*, 1880, vol. 171, part 2, p. Hastings: *Amer. J. Sc.*, 1888, ser. 3, vol. 35, p. 60.

rst surface are allowed to produce their effects at the second surface, sultant disturbance of the area $A\,B$ is identical with that which would lly follow from a rectilinear transference of the disturbances at points e area $a\,b$ to corresponding points on the area $A\,B$; and thus that the . Or, which represents the distance to which the front of disturb- nas travelled in the direction Or in a given interval of time, also sents the velocity of transmission of a ray of light in the same ion. Regarded from this point of view, the surface of disturbance ve-surface may be termed the *ray-surface*.

Ray-front.

rther, if a pencil of rays having OrR for axis starts simultaneously Rr, the front of the pencil at a certain epoch is a portion of the ray- e containing r, and at a subsequent epoch is a portion of the ray-

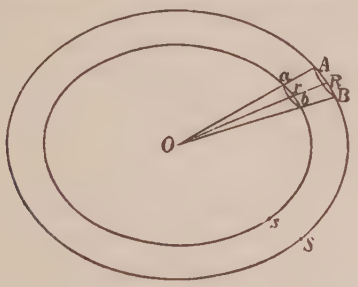

Fig. 2.

ce containing R: in the limiting case, where the pencil is of extremely angle, its front is in the tangent planes at r and R at successive s. The plane front, which thus belongs to an extremely small pencil ing a given ray, may be briefly denoted as the *ray-front* for that

ce the ray-surface retains a constant similarity of form and position, e ratio $OR : Or$ depends solely on the time, the tangent planes at l r are parallel.

en the ray-surface is not a sphere, the tangent plane at any point is neral inclined obliquely to the radius vector drawn to the point from igin, and a ray-front is then oblique to its corresponding ray.

e case of calcite, the ray-surface has two sheets and consists of a sphere and a spheroid.

ygens was thus led to the discovery that the laws of refraction in

the case of calcite are consistent with an undulatory theory, if the velocit of transmission of rays of light within this mineral are determined by sphere and a spheroid, touching each other in the axis of revolution of t latter.

If a line Or_2r_1 (Fig. 3), drawn from the common centre O, intersects t sphere and spheroid in r_2 and r_1 respectively, according to Huygens t velocity of transmission of one ray in the direction Or_2r_1 is measured Or_2, and of the other by Or_1.

For a single direction of Or_2r_1, namely that of the axis of revolution CO the two points r_2 and r_1 coincide, and the rays travel with equal velocity this direction is called the *optic axis* of the crystal.

Plane-polarisation of each of the refracted rays.

So far we have had regard merely to the relation of the two velociti to the direction of ray-transmission within the calcite-crystal. It did n escape the notice of Huygens, however, that each of the rays emerge from a crystal of calcite differs from ordinary light: the difference is o which he was unable to account for by his version of the undulato theory. It was not till more than a century afterwards (1808) that Mal made the accidental discovery that the same change in the character of t light may be induced by reflection from a plate of glass: to this alter tion, termed by Malus *polarisation*, the attention of physicists was large directed during the immediately succeeding years.

If a plate of calcite be placed with its faces perpendicular to an aerial transmitted plane-polarised ray, the latter is in general divided at the fir surface into two: the two rays travel through the plate in directions mutual inclined to each other, and, emerging from it, are transmitted through t air with the same direction as that of the original ray: each of the eme gent rays is plane-polarised, but the planes of polarisation of the rays ha not the same direction; they are, in fact, perpendicular to each othe Further, when the plate is turned round its normal through any angle, t plane of polarisation of each emergent ray is also displaced throu exactly the same angle: the direction of the plane of polarisation is th dependent on characters belonging to the plate itself: it is found to be i dependent of the direction of the plane of polarisation of the ray incide on the plate.

For one of the emergent rays, the line of transmission within the pla is continuous with the path of the ray before incidence and after eme gence: as the position of the plane of polarisation of the emergent ray independent of the thickness of the plate, we may reasonably assume,

re in the case of glass, that the ray transmitted *within the plate* is like- plane-polarised, and that the plane of polarisation during such trans- ion is identical in direction with that of the *emergent* ray, thus rotating the plate as the plate is turned round its normal.

s the emergent rays are indistinguishable from each other in charac- and only differ in the positions in space of their planes of polarisation, ay likewise assume that the second emergent ray is also transmitted in the plate as a plane-polarised ray, but with a direction of plane of risation perpendicular to that of the first.

will be found that the characters of rays which have been transmitted ugh a plate of calcite can be accounted for, if we imagine that in such dium a plane-polarised ray transmissible in any given direction has lane of polarisation in one or other of two rectangular positions, which nd on the crystal itself.

s in the case of air, glass, and analogous media, the periodic change of ther at every point of a plane-polarised ray transmitted within any fractive medium may be assumed to be dissimilarly symmetrical to perpendicular planes; but it may be remarked that it is only the *ether* which is assumed to be dissimilarly symmetrical in the ribution of its characters.[1]

plane of polarisation is related to the radius vector of the ray-surface.

nus[2] discovered that the direction of the plane of polarisation of any ransmitted within a crystal of calcite is determined by the direction the corresponding radius vector of the ray-surface: he showed that plane of polarisation for the ray Or_1 (Fig. 3) corresponding to the roid is always perpendicular to the plane Or_1C, which contains the direction and the optic axis, and for the ray Or_2 corresponding to the e is the plane Or_2C, which also contains the ray-direction and the axis: in other words, the plane containing the ray-direction and the axis is the plane of polarisation of the ray belonging to the sphere the transverse plane of the ray belonging to the spheroid.

above might have led to the recognition of the possible existence and the optical characters of biaxal crystals.

he above facts and reasoning were known to physicists before the exis- e of biaxal crystals had been discovered; further, the reasoning is y independent of the physical nature of the vibratory change which titutes light. We proceed to prove that though the geometrical

[1] See also pages 283, 355.

[2] *Mém. prés. à l'Institut:* Paris, 1811, vol. 2, p. 413

representation of the laws of transmission of light in biaxal crystals
suggested to Fresnel by ideas in which elasticity had a great p
the possible existence of such crystals, and the corresponding laws
transmission of light, might have been deduced from the above by a sim
generalisation, involving no reference either to the constitution of
luminiferous ether or to the nature of the physical change involved in
transmission of light; and further, that the step was so natural a on
take that the discovery of the true form of the wave-surface for bi
crystals could scarcely have been long avoided.

*Another mode of geometrically representing the characters of the ex
ordinarily refracted ray, by reference to the same spheroid, natur
presents itself.*

Draw OR_1 parallel to r_1V (Fig. 3), the tangent at r_1 to the ellipse

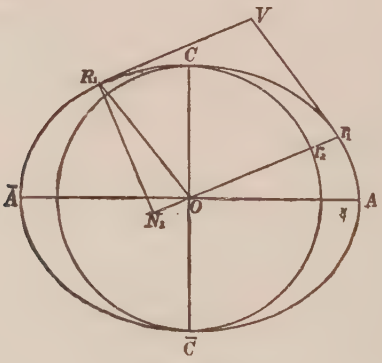

FIG. 3.

which the spheroid is cut by the plane r_1OC; OR_1 and Or_1 are said t
conjugate to each other, and the tangent at R_1 is parallel to Or_1. B
well-known property of the ellipse, the area of the parallelogram of w
OR_1 Or_1 are adjacent sides is constant, whatever the direction of Or_1: h
the area is $OA \cdot OC$; OA and OC being the principal axes of the ellipse,
therefore conjugate to each other. But if R_1N_1 is perpendicular to
meeting it in N_1, and is thus normal to the ellipse and therefore
to the spheroid at R_1, the area of the parallelogram is also $Or_1 \cdot R_1N_1$.

Hence $Or_1 = \dfrac{OA \cdot OC}{R_1N_1}$, whatever the direction of Or_1.

In other words, the *velocity* of a ray transmitted in the direction Or_1
be represented, not only by Or_1, but by the inverse of R_1N_1. But

line R_1N_1 determines the *plane of polarisation* of the ray Or_1, for as y stated, the line R_1N_1 is normal to that plane. Further, the same $_1N_1$ determines the *direction* of the ray, for the ray passes through O perpendicular to R_1N_1.

ce the direction, velocity and plane of polarisation of the ray Or_1 can represented by means of a single corresponding line R_1N_1, which is e normal to the spheroid and the ray.

s mode of representation naturally presents itself *as soon as the of polarisation is indicated by its normal*; in fact, any attempt to ent geometrically the observed facts of the double refraction of cal- most inevitably leads to it.

ame mode also suffices to represent the characters of the ordinarily refracted ray without necessitating the use of a second surface.

for any radius vector Or_1 of a spheroid there are always *two* normals spheroid which intersect it perpendicularly: one of them has just been ted, namely R_1N_1; the other is normal to the plane r_1OC, at the of the spheroid, and therefore always lies in the equatorial plane. eady stated, the *plane of polarisation* of the ray Or_2 is r_2OC or r_1OC: ne normal of its plane of polarisation thus lies in the equatorial plane, s normal both to the spheroid and the ray. Further, the intercept by the ray Or_2 upon this normal of the spheroid is OA, whatever the ion of Or_2r_1: hence, if the same law as before holds for the relation of locity to the intercept upon the normal of the spheroid, the velocity of y Or_2 is $\frac{OA \cdot OC}{OA}$, or OC; and this is exactly the velocity required.

nce the velocity and plane of polarisation of the ray Or_2 can likewise presented by means of a corresponding line which is at once normal to heroid and the ray: and this line indicates the plane in which the aving these characters will lie.

haracters of the refracted rays can be simply expressed by reference to the spheroid alone.

the characters of rays transmitted in various directions through a l of calcite may thus be simply expressed by means of a *single* sur- he spheroid. The relation of the optical characters of the crystal geometrical characters of the spheroid is as follows:—

every given point on a single surface, a spheroid, there in general ponds one ray: the *direction* of the ray is that of a diameter

intersecting perpendicularly the normal drawn at the point to spheroid ; the *velocity* of the ray is inversely proportional to the length the normal intercepted between the surface and the ray ; the *plane polarisation* of the ray is perpendicular to the same normal.

For points of the spheroid lying on the equatorial circle or at the e of the axis of revolution, the normal passes through the centre, and direction of the ray becomes indeterminate : such a point may be garded as the limiting case of a small circle ; and thus corresponds, no a single ray, but to an infinity of rays lying in a plane perpendicu to the normal, all transmitted with the same velocity, and all having same plane of polarisation.

In the case of singly refractive substances there is a spherical surfac reference for which the same general relations are true.

Generalisation.

But it immediately suggests itself that in the case of a crystal barytes, of which the morphological development and the physical cha ters are dissimilarly symmetrical to three rectangular planes, the sur of reference, if such a surface exists, is more likely to be an ellipsoid three unequal axes than an ellipsoid of which two axes are equal. In f the correspondence of the optical and the morphological symmetry crystals was announced by Brewster in 1819.

In the fourth Section are deduced the laws of transmission of ligh a crystal for which the surface of reference is an ellipsoid having th unequal axes ; starting with the hypothesis that the relations between geometrical characters of the surface of reference and the optical charac of the medium are identical with those which have just been found obtain when the surface of reference is either a spheroid or a sphere.

The Optical Indicatrix.

To the surface of reference the term *Optical Indicatrix* may be assig this suggestive term has the advantage of being equally applicable whe the surface of reference is an ellipsoid, a spheroid, or a sphere, and independent of all versions of the undulatory theory ; the adjectival p may be omitted when the term Indicatrix involves no ambiguity. Indicatrix is identical in form with the *ellipsoid of elasticity* of var authors, the *ellipsoid of polarisation* of Cauchy, the *ellipsoid of indic* Mac Cullagh, and the *index-ellipsoid* of Liebisch.

Relation of the optical indicatrix to the general symmetry of the crystal.

In regard to the arrangement of its faces, every crystal is found to belong to one or other of six types of symmetry, distinguished as cubic, tetragonal, hexagonal, ortho-rhombic, mono-symmetric, and anorthic: further, it has been demonstrated by the mathematician that the types of crystalline symmetry thus met with are precisely those which are presented by systems of planes of which the relative positions can be expressed by means of whole numbers, a law to which the faces of crystals are found to conform. Further, we are led by experiment to the induction that a type of symmetry is such, not only for the arrangement of the faces of a crystal, but for all the physical characters: the planes of symmetry characteristic of the types are thus planes of *general symmetry*.

On the other hand, a plane may be one of symmetry for a particular character without being a plane of general symmetry of the crystal: the axe is thus not necessarily determinable from the symmetry of the crystal with respect to a single character. For example, a crystal may have the six faces of a cube and really belong, not to the cubic, but to the tetragonal, or even the ortho-rhombic type; observation of some character other than the geometrical being thus necessary to the distinction: again, a plane inclined at any angles to the planes of general symmetry of a cubic crystal, and any plane containing the morphological axis of a tetragonal or hexagonal crystal, is a plane of symmetry for the changes produced by dilatation on change of temperature, and is generally not a plane of symmetry for the facial arrangement.

The above induction requires a plane of general symmetry to be a plane of symmetry of every indicatrix: on the other hand, a plane of symmetry of an indicatrix is not necessarily a plane of general symmetry of the crystal.

Hence, if the most general form of the indicatrix be an ellipsoid, it will follow that in the case of an ortho-rhombic crystal the axes of any indicatrix must coincide with the three axes of general symmetry. For a tetragonal or hexagonal crystal, the symmetry of the indicatrix with respect to the general planes of symmetry requires two of the axes of the ellipsoid to be equal, and the ellipsoid to be one of revolution about the morphological axis. For a cubic crystal, the symmetry of the indicatrix with respect to the general planes of symmetry necessitates the equality of all the axes of the ellipsoid, and the surface becomes a sphere.

The above is true for all colours of the light, though the relative magnitudes of the axes, both for the general ellipsoid and the ellipsoid of revo-

lution, may vary with the colour : further, it is true for all temperatur of the crystal consistent with the stability of the structure, for a plane general symmetry must retain that character between the assumed lim of temperature.

In the case of a mono-symmetric crystal, the induction still requires t plane of general symmetry to be a plane of symmetry of the indicatrix f all colours of light and for all temperatures consistent with crystalli stability; but the positions and dimensions of the two axes of the ellipsc lying in the plane of general symmetry are otherwise independent of t latter, and will in general vary both with the colour of the light and t temperature of the crystal.

And in the case of an anorthic crystal, in which there is a centre, but plane, of general symmetry, the positions and dimensions of all three re tangular axes of the indicatrix corresponding to a given colour or temper ture are free from limitations by a plane of general symmetry, and w likewise vary both with the colour of the light and the temperature which the determinations are made.

Section III.—Naturalness of the method.

Objections.

To the above reasoning, by which it is sought to prove that in the ca of calcite the reference of the two sheets to the spheroid alone is o which it is natural to make, and not a mere geometrical artifice only be discovered after the truth of the generalisation has been establishe it may be objected that the reference would in such case have been ma long before the present century. It must be remembered, however, th the consequent generalisation would have been a barren speculation at time when the polarisation of light by reflection was still undiscover (1808), and the optical characters of most doubly refractive crystals we still beyond the powers of observation; indeed, it was not till the deca 1810-20 that any series of numerical data were available for the testing

theory: even the accuracy of the construction given by Huygens for determination of the directions of the refracted rays in calcite was predited by most physicists at the beginning of this century.

But it may be fairly objected that if the above reference and generalisa- be natural, the discovery of the process would have preceded the elopment of any elastic theory of double refraction. And, in fact, it really by a process of generalisation that Fresnel's discovery of the form of the ray-surface for biaxal crystals was made. When the ve argument was written, the detailed history of Fresnel's theory had come to the notice of the author: as the facts are not generally wn, and have an important bearing on the true significance of the tic theory of double refraction, it becomes desirable to explain the ition.

The development of Fresnel's theory.

Fresnel's celebrated memoir on Double Refraction was not printed till 7: in that year, and before the issue of the memoir, Fresnel died at early age of 39, after years of illness. In the memoir are incorporated ers submitted to the Academy at different dates in the years 1821 1822, and it occupies no less than 132 pages of large size. For the of brevity, Fresnel made many omissions from the papers as origin- submitted to the Academy, and for the sake of clearness adopted a thetic mode of treatment: the result is that the memoir as printed es no clue to the real order of discovery, and the reader is apt to infer Fresnel discovered the true form of the ray-surface *a priori* by means equations relative to the elastic forces evoked by the disturbance of incompressible elastic ether. The following statement by Aldis[1] exem- es this, which is still a very general impression:—

Fresnel's theory is undoubtedly not a sound dynamical theory. It however, the great merit of representing accurately the facts of ble refraction as far as experiment at present has tested them, and in instance has led to the discovery of facts (the conical refractions) pre- isly unobserved. Probably, when the Newton of Physical Optics has ceeded in linking together all the phenomena of Light into one tinuous chain, the name of Fresnel will yet be remembered with a erence akin to that which astronomers feel for Copernicus and Kepler."

The real order of development was of course known to some of Fresnel's

[1] W. S. Aldis. *A Chapter on Fresnel's Theory of Double Refraction.* Cambridge, , p. 26.

contemporaries, but to the next generation it was a mystery; it was till forty years after Fresnel's death that the mystery was dispelled by publication of the original memoirs, which had been carefully preser in the family. Verdet,[1] one of the editors of Fresnel's collected pap makes the following remarks:—

"It may seem odd that reasoning which is incomplete and inexac two points should have for result one of the best confirmed of the Law Nature. But we have seen that this law became manifest to Fresnel as result of a generalisation quite similar to the generalisations which have to most great discoveries. When he wished afterwards to account for law by a mechanical theory, it is not astonishing that he should h led the theory, perhaps unwittingly, towards the end which he alre knew of, and that, in his choice of hypotheses, he should have been termined, less by their intrinsic probability, than by their agreement what he was justified in believing to be true. We have seen some tr of the progress of his ideas in the marginal notes which he had adde the manuscript of memoir No. 38, a memoir here printed for the time. In the later memoirs we find nothing but the explanation different forms, of the mechanical theory by which he tried to dem strate *a posteriori* the laws which direct intuition had revealed to him.

After this clear statement on the part of his editor, it is obvious Fresnel's theory of double refraction, however ingenious, has no cl to credit for its predictions: the latter are really a direct consequ of the generalisation which had preceded the theoretical developmer the vibratory properties of an elastic but incompressible ether.

Preliminary attempts at generalisation.

The first attempt at the generalisation of Huygens' construction had gested a sphere combined with a concentric ellipsoid having three une axes as the most general form of ray-surface: this assumed that in most general case one of the rays obeys the ordinary laws of refractio

It was found, however, that the refraction of the second ray as exp mentally determined is inconsistent with an ellipsoidal form of surface. Nor would such a combination of ray-surfaces account for optical characters of a biaxal crystal: for if a concentric sphere ellipsoid meet each other, they must either touch at the extremities principal diameter, or intersect in two curves; in the former case t would be only one direction of equal ray-velocity; in the latter case

[1] *Œuvres Complètes d'A. Fresnel*: Paris, 1868, vol. 2, p. 327.

aracter would belong to every diameter which passes through the curves intersection, and thus to an infinity of lines lying on the surface of a ne.

In 1819 Biot made the important discovery that the results of optical asurement are consistent with two empirical laws, both of them ched by processes of generalisation: in combination with the assump- n that one of the rays obeys the laws of ordinary refraction, they npletely express the polarisation and velocity of the second ray in terms its direction in the crystal.[1]

st law. We have seen that in the case of a uniaxal crystal, two rays emitted along any given direction had been shown by Malus to have ir planes of polarisation respectively coincident and at right angles with plane containing the ray-direction and the optic axis: from this Biot led by generalisation to the discovery that in the case of a biaxal planes of polarisation are the internal and external bisectors of the le between the two planes which contain the ray-direction and pass h through one of the optic axes.

nd law. In the case of a uniaxal crystal, if v_1 and v_2 be the velocities ransmission of the two rays transmissible in a direction inclined at an le σ to the optic axis, it follows from Huygens' construction that the e $\left(\frac{1}{v_1^2}-\frac{1}{v_2^2}\right):\sin^2\sigma$ is constant for all directions: noticing this, Biot ied by generalisation to the discovery that in the case of a biaxal tal the ratio $\left(\frac{1}{v_1^2}-\frac{1}{v_2^2}\right):\sin\sigma_1\sin\sigma_2$ is constant, σ_1 σ_2 being the incli- ons of the ray to the optic axes.

he second law, combined with the assumption that the velocity of one be rays is independent of its direction, leads to a surface of the fourth ee, tangent to a concentric sphere at the ends of two diameters, as the surface corresponding to the second ray.

or, let $l\ 0\ n\ \bar{l}\ 0\ n\ \lambda\ \mu\ \nu$ be direction-cosines of the optic axes and of the nd ray respectively (Fig. 9):

then $\cos\sigma_1 = l\lambda + n\nu$, and $\cos\sigma_2 = -l\lambda + n\nu$.

r be the variable velocity of the second ray, and a be the constant city of the first ray, it follows from the above law that

$$\frac{1}{r^2}-\frac{1}{a^2} = k\sin\sigma_1\sin\sigma_2,$$

re k is a constant.

[1] *Mémoires de l'Acad. de l'Institut de France*, 1820, vol. 3, pp. 228, 233.

Hence $\frac{1}{k^2}\left(\frac{1}{r^2}-\frac{1}{a^2}\right)^2=(1+\cos\sigma_1)(1-\cos\sigma_1)(1+\cos\sigma_2)(1-\cos\sigma_2)$

Substituting the values of $\cos\sigma_1$ and $\cos\sigma_2$, and writing

$$\lambda=\frac{x}{r},\ \mu=\frac{y}{r},\ \nu=\frac{z}{r},$$ we have for the equation of

second ray-surface,

$$\frac{1}{k^2}\left(1-\frac{r^2}{a^2}\right)^2=(lx+nz+r)(lx+nz-r)(lx-nz+r)(lx-nz-$$

an equation of the fourth degree. The second ray-surface meets
first ray-surface ($r=a$) at the intersection of the latter with the
planes, $lx\pm nz\pm a=0$, and these planes are tangent to the sphere at
extremities of the optic axes: hence the two surfaces are, tangent to e
other at the same points.

The history of the ray-surface.

At the advent of Fresnel, the emissive theory of light still held alu
undisputed sway in the scientific world, notwithstanding the interfere
discoveries which had been made by Dr. Thomas Young, many y
before. Convinced that the true explanation of interference was furnis
by the undulatory theory, Fresnel devoted himself with ardour to
theoretical and experimental development, in which he had to sus
the attacks of Laplace, Poisson and Biot, who were firm believers in
truth of the older theory. After successful explanation of diffraction
of the polarisation-colours of thin plates on the undulatory hypoth
Fresnel in 1821 attempted to solve the problem of double refraction.

Young had already suggested (12 Jan. 1817) that the vibrations of
luminiferous ether are motions transverse to the ray, and had compa
the transmission of a ray of light to the transmission of a transv
vibration along a stretched cord:[1] Fresnel himself, in conjunction v
Arago, had shown experimentally that two rays polarised in perpendic
planes are incapable of mutual interference, thus confirming the idea
the motion of the ether of a plane-polarised ray is wholly perpendicula
the direction of transmission, so long, at least, as the medium is sir
refractive.

Fresnel therefore suggested that the difference of velocities of the
rays transmissible in the same direction in a doubly refractive medium
be due to differences of elastic force evoked by equal displacement
ethereal particles in different directions: the same suggestion had b

[1] *Œuvres Complètes d'A. Fresnel:* vol. 1, p. 634; vol. 2, p. 742.

blished by Young[1] in 1809, before the idea of transverse vibrations d presented itself. Assuming that the ethereal elasticity, in the e of a uniaxal crystal, is the same for all directions of displacent perpendicular to the optic axis, and different from that evoked by placements parallel to it, and that the vibrations are always *perpendiarly* transverse to the ray[2] (an assumption he departed from later), esnel showed that the vibration of the ordinary ray must be perpendicular its plane of polarisation; for in this case the vibration, being always pendicular to the optic axis, evokes an elastic force of which the gnitude is the same whatever the direction of the ray. The vibration the extraordinary ray being likewise assumed to be perpendicular both the ray and its plane of polarisation, the elastic force evoked by it will y with the direction of the vibration, and thus the velocity will depend the inclination of the ray to the optic axis.

But Fresnel[3] soon saw that, if such an explanation is true, neither of the s transmitted in a biaxal crystal can have a velocity independent of direction, for in such a crystal there is no direction such that the ical characters are the same in all directions equally inclined to it. ving submitted this inference to the test of experiment, Fresnel nounced its confirmation in September 1821, thus completely upsetting ideas which then prevailed as to the forms and relations of the two e-surfaces of a biaxal crystal.

urther developing the theory, Fresnel[4] showed (November 19, 1821) t if the ethereal elasticity be proportional to the square of the velocity, in the case of the longitudinal vibrations of an elastic medium, a face such that any radius vector represents the square of the elastic e evoked by a unit displacement in its direction is, in the case of a axal crystal, a spheroid (distinct from the spheroidal ray-surface itself), ie *double refraction is small*. Conversely, lines measured in a direction pendicular to a diametral plane of the auxiliary spheroid, and having ths equal to the maximum and minimum radii vectores of the section, ld *approximately* represent the velocities with which vibrations, parallel hose radii vectores, would be transmitted along the normal of the plane. then suggested itself to Fresnel that an ellipsoid with three unequal might be a more general form of this surface of elasticity, and that same construction might hold for the determination of the *approxi*e velocities of the rays having a given direction. The two circular

[1] *History of the Inductive Sciences*; by Whewell: London, 1857, vol. 2, p. 329.
[2] *Œuvres Complètes d'A. Fresnel*: vol. 2, p. 281.
[3] *Ibid.*, p. 257.
[4] *Ibid.*, pp. 285, 304, 306.

sections of the ellipsoid immediately account for the existence of t optic axes, for there will be only one velocity of transmission along th normals: again, the planes of polarisation for a given ray-direction are right angles, for the corresponding vibrations are parallel to the axes of diametral section perpendicular to the common direction. It remained discover whether the two empirical laws established by Biot were geor trically consistent with the form of ray-surface suggested by the above co struction: and this being found by Fresnel to be very approximately ease, the true form of ray-surface was at last determined.

In this way, however, it presented itself as an approximation, true o when the double refraction is small: it was not till later (26 Nov. 18 that the hypothetical form of the surface of elasticity[1] was changed from ellipsoid to a surface of the fourth degree: with this hypothesis the r surface already obtained as an approximation appeared as the true r surface, independently of the amount of the double refraction of medium.

It follows from the above that Fresnel's discovery of the form of ray-surface for biaxal crystals was really arrived at by a geometri generalisation of Huygens' ray-surface for uniaxal crystals, and that geometrical relation used in the generalisation was suggested by the c ception of a plane-polarised ray as due to vibratory motion perpendicu to its direction.

The true nature of the luminiferous ether.

Notwithstanding the success with which so many optical properties h been explained on the hypothesis that light is a vibratory motion of elastic ether of which the effective density depends on the permea matter, it would be wrong to infer that light is *actually* due to suc vibratory motion. It is conceivable that other hypotheses may likew lead to similar results: and, indeed, any other quantities about which same general assertions may be made and which obey the same mat matical laws will satisfy the equations and furnish other analogies. fact, Gibbs[2] has lately shown that the equations which result from the l version of the elastic theory have a corresponding electrical interpretati

Again, according to Clerk Maxwell, light is not a vibratory motion to elasticity, but an electro-magnetic vibration (whatever that may be); a he showed that the velocity of transmission of electro-magnetic action

[1] *Œuvres Complètes d'A. Fresnel:* vol. 2, p. 338.
[2] *Phil. Mag.* 1889, ser. 5, vol. 27, p. 238.

ether is identical with that of light. The recent experiments of Hertz placed the undulatory character of electro-magnetic radiation beyond region of hypothesis, and it is now experimentally established that ro-magnetic waves and light-waves differ only in length; while the -length for sodium-light is ·000589 millimetres, the short electro-netic waves produced by Hertz had still a length of 2 metres. On ther hand, it seems that electro-magnetic actions are inexplicable as vibratory motion of an elastic ether: Professor Fitzgerald,[1] who has much thought to the mechanical representation of the ether, points hat "if magnetic forces are analogous to the rotation of the elements wave, an ordinary solid cannot be analogous to the ether, because atter may have a constant magnetic force existing in it for any length ne, while an elastic solid cannot have continuous rotation of its nts in one direction existing within it."

The educational difficulty.

king everything into consideration, it seems undesirable, from the y educational point of view, to continue such a synthetic mode of ment as was adopted by Fresnel in the memoir of 1827. The t of the optical properties of crystals is so extensive that it is un-factory to make all the laws appear to depend upon an hypothesis of uth of which we are not convinced: otherwise it becomes necessary to keep the student in ignorance of the doubts as to the truth of ypothesis, or to raise a feeling of distrust as to the accuracy of every tion therefrom. It would seem better to develop the subject by s of analogy and experiment, and to assign a subordinate importance e mechanism of the ether.

Three other modes of generalisation.

addition to the method explained in the last Section there are at least others by which a generalisation of Huygens' construction may rived at. One of them depends on the fact that the wave-surface is velope.

Of_2f_1 (Fig. 4) be any line intersecting the sphere and spheroid, ve consider a section of the surface by a plane containing the optic and the given line, it follows that $Of_1 = \frac{OA \cdot OC}{OP_1}$, where OP_1 is per-

[1] *Nature*, 1890, vol. 42, p. 173.

pendicular to Of_1, and that Of_2 is equal to OC or $\frac{OA \cdot OC}{OA}$. Hence
lines Of_1 Of_2, which represent the velocities of the wave-fronts $f_1 r_1$ $f_2 g$
inversely proportional to the lines OR_1 and OA, or to the axes of
ellipse in which the spheroid is intersected by a plane parallel to
wave-fronts $f_1 r_1 f_2 g$.

Huygen's wave-surface is thus the envelope of planes which
parallel to the same central section of the spheroid, and pass thro
those two points on the common central normal which are distant
the centre by lengths inversely proportional to the axes of the sec
On generalising this result, by the substitution of an ellipsoid with t
unequal axes for the spheroid, Fresnel's wave-surface is obtai

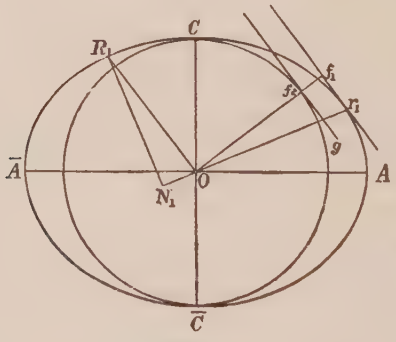

FIG. 4.

this generalised construction is virtually identical with the one empl
by Fresnel in the memoir of 1827; the only difference being tha
used the inverse of the ellipsoid instead of the ellipsoid itself.

The two other geometrical constructions, which present themselve
the generalisation of Huygens' construction, only appear indirectly:
depend on corresponding relations between the rays or ray-fronts and
polar reciprocal of the indicatrix.

Advantages of the method here suggested.

There are thus two purely geometrical processes, which *directly*
sent themselves for trial as being possible modes of representing
optical characters of those crystals which belong to a lower type of ge
symmetry than the uniaxal; both lead to identical results. If we

npelled to select one or other of these processes for use as an educa-
nal instrument, there can be no doubt on which the choice must fall.
The method based on an envelope is so far wanting in simplicity that
esnel himself gave no rigorous solution of the equations: this was sup-
ed by Ampère, and it was not till even a decade later that the less
nplicated mode of elimination, now generally given, was invented by
chibald Smith. Further, the construction yields the wave-surface in
h a way that its singularities are not obvious, and were only remarked
Sir William Hamilton several years after Fresnel's death.

On the other hand, as was shown in Section II, the geometrical basis
re advocated naturally suggests itself as soon as any attempt is made
represent geometrically the observed optical properties of uniaxal
stals: we shall further show that it readily furnishes the equation of
ray-surface without demanding any knowledge of the differential
culus or any determination of maxima and minima; that it im-
diately suggests all the singularities of the ray-surface; and that,
fact, most optical problems are reduced to a form in which their solu-
can be effected by the elementary geometry of the ellipsoid. We may
that the employment of an additional ellipsoid, the polar reciprocal of
first, is rendered unnecessary, and a continual source of confusion to
student is thereby removed.

CTION IV.—DEDUCTION OF THE OPTICAL CHARACTERS CORRESPONDING TO AN ELLIPSOIDAL INDICATRIX.

General Relation.

The characters of a ray of plane-polarised homogeneous light transmitted hin a medium are indicated by geometrical characters at a corresponding nt on an ellipsoid; the direction of the ray is that of a diameter in-secting perpendicularly the normal drawn to the ellipsoid at the cor-ponding point; the velocity is inversely proportional to the length of the mal intercepted by the ray; the plane of polarisation is perpendicular to normal.

t is required to deduce the relations of the optical characters for erent directions in the medium,

The order of deduction is as follows:—

Arts. 1-15 relate to *rays* in general; Arts. 16-20, to the particular ray directions here termed *bi-radials*; Arts. 21-40, to *ray-fronts* in general; Arts. 41-44, to the particular front-normals here termed *bi-normals*; Arts. 45-52, to the *bi-radial* and *bi-normal cones*.

1. The construction of the ray-surface.

Let $a^2x^2 + b^2y^2 + c^2z^2 = 1$ be the equation of the indicatrix; a, b, being in descending order of magnitude: O the centre of the indicatrix; x' y' z' the co-ordinates of R, a point on the indicatrix: NOr a line intersecting the normal RN perpendicularly (Fig. 5).

According to the above relation, the *direction of the ray* corresponding to the point R of the indicatrix is given by the line NOr, the *velocity* of the ray is measured by $\frac{1}{RN}$: the *plane of polarisation* is perpendicular to RN.

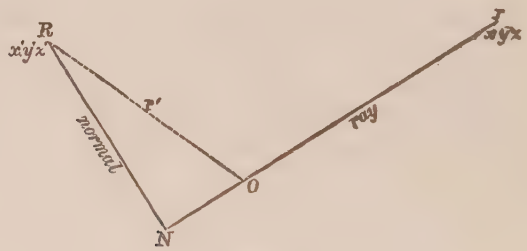

Fig. 5.

Take the length Or equal to $\frac{1}{RN}$: the locus of the points r, corresponding to all positions of R on the indicatrix, will be the *ray-surface* of the medium for the given simple colour: since the velocity of transmission of a ray of light of that colour along any radius vector of the surface is measured by the length of the radius vector.

The propositions of the present Section are stated in a form which is independent of any particular version of the undulatory theory: it may be remarked, however, that according to the latest version of the elastic theory, RN is the *direction of vibration* for the ray Or; according to Fresnel's version of the elastic theory and the present statement of the electro-magnetic theory, the direction of vibration is RO.[1]

[1] *Philos. Magazine*, 1888, ser. 5, vol. 26, p. 528: *Electricity and Magnetism*, by *J. Clerk Maxwell*: Oxford, 1881, vol. 2, p. 404; *Nature*, 1890, vol. 42, p. 174.

2. *The symmetry of the ray-surface.*

From the mode of construction, it is evident that the planes of symmetry f the indicatrix are also planes of symmetry of the ray-surface.

3. *The sections of the ray-surface by the planes of symmetry.*

The section of the ray-surface by each plane of symmetry consists a circle and an ellipse; the radius of the circle is the inverse of that is of the indicatrix which is perpendicular to the plane of the section; e ellipse is similar and similarly situated to the section of the indicatrix the same plane. This may be proved as follows:—

Let $AO\bar{A}$, $BO\bar{B}$, $CO\bar{C}$, be the principal axes of the indicatrix, and $A = \frac{1}{a}$, $OB = \frac{1}{b}$, $OC = \frac{1}{c}$: it is required to determine the section of e ray-surface by one of the planes of symmetry, say AOC (Fig. 6).

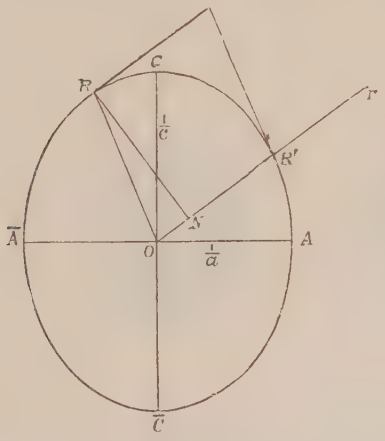

FIG. 6.

a. By considering a series of points in a small ring surrounding the int B on the indicatrix, it is seen that in the limit the point B itself rresponds, not to one ray, but to an infinity of rays, all lying in the ane AOC; for the axis OB is the normal of the indicatrix at B, and ersects perpendicularly all radii vectores of the indicatrix which lie in the ane AOC. Further, the length of the normal at B intercepted by each the rays is OB: hence the velocity of each ray corresponding to the int B is $\frac{1}{OB}$ or b, and a circle of radius b, situated in the plane AOC, on the ray-surface (Fig 7c).

b. For a point R on the indicatrix, lying in the plane AOC, the norm RN of the indicatrix lies in the plane AOC, and is also normal at the poi R to the elliptic section $AC\bar{A}\bar{C}$: if ONr is perpendicular to RN a $Or = \frac{1}{RN}$, Or is by construction a radius vector of the ray-surface (Fig. 6

If R' be a point in which Or intersects the indicatrix, OR and OR' a conjugate to each other, for Or being perpendicular to RN is parallel the tangent of the ellipse at R: hence the product $OR' \cdot RN$, whi measures the area of the parallelogram of which OR OR' are adjace sides, is constant, and equals the product $OA \cdot OC$ or $\frac{1}{ac}$.

Hence, $Or = ac\ OR'$

The locus of r is thus an ellipse, similar and similarly situated to t ellipse $AC\bar{A}\bar{C}$, and having ac times its dimensions: its semi-axis in t line OA will thus be c, and its semi-axis in the line OC will be a (Fig. 7

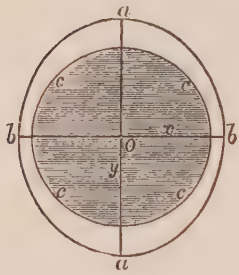

Fig. 7A.

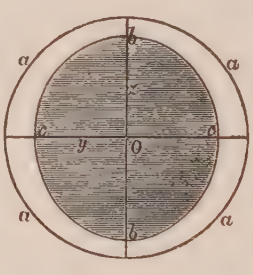

Fig. 7B.

Fig. 7C.

c. To any ray lying in the plane AOC no other normal of the indicatr is perpendicular: hence no radius vector of the ray-surface, other th the above, lies in the plane AOC; the circle $z^2+x^2=b^2$ and the ellip $\frac{z^2}{a^2}+\frac{x^2}{c^2}=1$ are thus the only curves of intersection of the ray-surface wi the plane AOC (Fig. 7c).

d. And the section of the ray-surface by the plane of symmet BOA is a circle $x^2+y^2=c^2$ and an ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ (Fig. 7A).

e. Similarly, the section of the ray-surface by the plane of symmet COB is a circle $y^2+z^2=a^2$, and an ellipse $\frac{y^2}{c^2}+\frac{z^2}{b^2}=1$ (Fig. 7B).

On consideration of the relative magnitudes of a, b, c, it will be seen in one of the planes of symmetry the circle falls wholly within the ...se; in another, the ellipse falls wholly within the circle; in the third, ...aining the longest and shortest axes of the indicatrix, the circle and ...se intersect each other in four points lying at the extremities of two ...meters $s_1 \bar{s}_1$, $s_2 \bar{s}_2$.

For any direction of ray lying in a plane of symmetry, there are ... two possible directions for the plane of polarisation, perpendicular to ... other; and in general each plane of polarisation corresponds to a ...rent velocity of transmission in the given direction. For two directions ...the ray, those of the diameters $s_1 s_1$, $s_2 s_2$, the two velocities are equal (... 7c).

... $x\,0\,z$ be the co-ordinates of one of the points s, we have

$$\frac{x^2}{c^2}+\frac{z^2}{a^2}=1 \text{ and } x^2+z^2=b^2\text{: hence}$$

$$\frac{x^2}{c^2(a^2-b^2)}=\frac{z^2}{a^2(b^2-c^2)}.$$

... $\lambda\,0\,\nu$ be the direction-cosines of a diameter Os, the relation may also ...written as

$$\frac{a^2\lambda^2}{b^2-a^2}+\frac{c^2\nu^2}{b^2-c^2}=0.$$

...he angle s_1OC is given by the relation

$$\tan s_1OC=\frac{c\sqrt{(a^2-b^2)}}{a\sqrt{(b^2-c^2)}}=\frac{\sqrt{\left(\frac{1}{a^2}-\frac{1}{b^2}\right)}}{\sqrt{\left(\frac{1}{b^2}-\frac{1}{c^2}\right)}}.$$

... In each of the planes OBC OCA OAB a plane-polarised ray is ... transmissible with the velocity a b or c respectively, whatever its ...ction in the plane: hence, by the general principle of undulations, ... refraction of these rays by a surface perpendicular to the symmetral ...e will be ordinary, and the index of refraction will be $\frac{v}{a}$, $\frac{v}{b}$, $\frac{v}{c}$ respec-...y; v being the constant velocity of transmission in the other ...ium. If $\alpha\beta\gamma$ be the values of the index of refraction of the rays in ... of the above planes, for which the index of refraction is independent ...he direction, we must have

$$a:b:c=\frac{1}{\alpha}:\frac{1}{\beta}:\frac{1}{\gamma};$$

$\alpha\beta\gamma$ are termed the *principal indices of refraction*.

4. *Given the co-ordinates $x'y'z'$ of R, to find the velocity r of corresponding ray Or.*

The velocity of the ray Or is measured by $\frac{1}{RN}$ (Fig. 5) : but being the normal of the indicatrix at the point R and perpendicular to by construction is equal to the perpendicular drawn from the origin the plane $a^2x'x + b^2y'y + c^2z'z = 1$ which touches the indicatrix at R. p be the length of this perpendicular

$$\frac{1}{p} = \sqrt{(a^4x'^2 + b^4y'^2 + c^4z'^2)}.$$

Hence $$r^2 = \frac{1}{RN^2} = \frac{1}{p^2} = a^4x'^2 + b^4y'^2 + c^4z'^2 :$$

r being the length of that radius vector of the ray-surface which corresp to the point R of the indicatrix.

5. *Given the co-ordinates $x'y'z'$ of R, to find the direction-cosine the normal of the plane of polarisation of the corresponding ray Or.*

The plane of polarisation, being perpendicular to the normal RN (5), is parallel to the tangent plane of the indicatrix at the point R ; equation of the tangent plane may be written in the form

$$pa^2x'x + pb^2y'y + pc^2z'z = p.$$

Hence the direction-cosines of the normal of the plane of polarisa are

$$pa^2x',\ pb^2y',\ pc^2z' ;$$

$$\text{or}\quad \frac{a^2x'}{r},\quad \frac{b^2y'}{r},\quad \frac{c^2z'}{r} :$$

where $r^2 = a^4x'^2 + b^4y'^2 + c^4z'^2$.

6. *Given the co-ordinates $x'y'z'$ of R, to find the direction-cosine the corresponding ray Or.*

The direction-cosines of RN (Fig. 5) being $\frac{a^2x'}{r}, \frac{b^2y'}{r}, \frac{c^2z'}{r}$, those of OR being $\frac{x'}{r'}, \frac{y'}{r'}, \frac{z'}{r'}$, where $r' = OR$, the direction-cosines of a line perpendicular to both RN and OR, and therefore to the p RON and all lines therein, are given by the equations:—

$$hx' + ky' + lz' = 0$$
$$ha^2x' + kb^2y' + lc^2z' = 0 :$$

hence

$$\frac{h}{y'z'(b^2 - c^2)} = \frac{k}{z'x'(c^2 - a^2)} = \frac{l}{x'y'(a^2 - b^2)}.$$

$\lambda\,\mu\,\nu$ be the direction-cosines of any line whatsoever in the same plane

$$\lambda h+\mu k+\nu l=0$$

$$\text{or } \lambda y'z'(b^2-c^2)+\mu z'x'(c^2-a^2)+\nu x'y'(a^2-b^2)=0.$$

t the line $\lambda\,\mu\,\nu$ coincide with Or, in which case it is at right angles to
N, of which the direction-cosines are $\frac{a^2x'}{r}, \frac{b^2y'}{r}, \frac{c^2z'}{r}$; we thus have

$$\lambda a^2x'+\mu b^2y'+\nu c^2z'=0.$$

termining the ratios $\lambda:\mu:\nu$ from the last two equations, we get

$$\frac{\lambda}{A}=\frac{\mu}{B}=\frac{\nu}{C}$$

ere

$$\begin{aligned}
A &= x'z'^2c^2(c^2-a^2) - x'y'^2b^2(a^2-b^2) = x'(r^2-a^2),\\
B &= y'x'^2a^2(a^2-b^2) - y'z'^2c^2(b^2-c^2) = y'(r^2-b^2),\\
C &= z'y'^2b^2(b^2-c^2) - z'x'^2a^2(c^2-a^2) = z'(r^2-c^2);
\end{aligned}$$

$$\text{or } \frac{\lambda}{x'(r^2-a^2)}=\frac{\mu}{y'(r^2-b^2)}=\frac{\nu}{z'(r^2-c^2)}$$

in which $r^2=a^2x'^2+b^2y'^2+c^2z'^2$, as before.

ese equations determine the direction-cosines $\lambda\,\mu\,\nu$ of the ray Or corponding to the point $x'y'z'$ or R.

7. The equation of the ray-surface.

ne co-ordinates of r being $x\,y\,z$, we have $x=\lambda r$, $y=\mu r$, $z=\nu r$: $\lambda\,\mu\,\nu$ g the direction-cosines of the ray Or.

ence, substituting these values in the last set of equations,

$$\frac{x}{x'(r^2-a^2)}=\frac{y}{y'(r^2-b^2)}=\frac{z}{z'(r^2-c^2)}.$$

ch of these fractions is equal to

$$\frac{a^2x\frac{x}{r^2-a^2}+b^2y\frac{y}{r^2-b^2}+c^2z\frac{z}{r^2-c^2}}{a^2xx'+b^2yy'+c^2zz'}.$$

the denominator of the last expression is zero; for by construction line Or, of which the direction-cosines are $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, is perpendicular
N, of which the direction-cosines are $\frac{a^2x'}{r}, \frac{b^2y'}{r}, \frac{c^2z'}{r}$ (Art. 5).

ence the numerator is also zero; for the fractions equivalent to the ression are never all of them indeterminate, and are never infinite.

$$\text{Thus } \frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0;$$

this is the equation of the ray-surface, for it expresses a relati between the co-ordinates $x\,y\,z$ of any point r lying in it.

The equation may also be written in the form

$$\frac{a^2x^2}{r^2-a^2}+x^2+\frac{b^2y^2}{r^2-b^2}+y^2+\frac{c^2z^2}{r^2-c^2}+z^2=r^2,$$

or

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2}=1.$$

8. *Given $\lambda\,\mu\,\nu$, the direction-cosines of a line of transmission, find $r_1\,r_2$ the velocities of the corresponding rays Or_1 Or_2.*

Substituting the values $x=\lambda r$, $y=\mu r$, $z=\nu r$ in the equation of the r surface, we have

$$\frac{a^2\lambda^2}{r^2-a^2}+\frac{b^2\mu^2}{r^2-b^2}+\frac{c^2\nu^2}{r^2-c^2}=0:$$

and, multiplying out,

$$r^4(a^2\lambda^2+b^2\mu^2+c^2\nu^2)-r^2\{a^2(b^2+c^2)\lambda^2+b^2(c^2+a^2)\mu^2+c^2(a^2+b^2)\nu^2\}+a^2b^2c^2=0$$

This being a quadratic equation in r^2, there are in general, for giv values of $\lambda\ \mu\ \nu$, two solutions, say $r_1^2\ r_2^2$, and thus two velocities of tra mission in the given direction.

A geometrical solution is given in Art. 15.

The above equation may sometimes be conveniently written in the fo

$$\frac{\lambda^2}{\frac{1}{r^2}-\frac{1}{a^2}}+\frac{\mu^2}{\frac{1}{r^2}-\frac{1}{b^2}}+\frac{\nu^2}{\frac{1}{r^2}-\frac{1}{c^2}}=0.$$

9. *Given $\lambda\ \mu\ \nu$, the direction-cosines of a line of transmission, to f the co-ordinates $x_1'\ y_1'\ z_1'$, $x_2'\ y_2'\ z_2'$ of the points $R_1\ R_2$ of the indicatrix wh correspond to the rays Or_1 Or_2 respectively.*

Having found r_1^2 and r_2^2, as indicated in the last Article, the ordinates $x_1'\ y_1'\ z_1'$, $x_2'\ y_2'\ z_2'$ are given by the equations (Art. 6):—

$$\frac{x_1'}{\frac{\lambda}{r_1^2-a^2}}=\frac{y_1'}{\frac{\mu}{r_1^2-b^2}}=\frac{z_1'}{\frac{\nu}{r_1^2-c^2}}=L_1\ (\text{say}),$$

and

$$\frac{x_2'}{\frac{\lambda}{r_2^2-a^2}}=\frac{y_2'}{\frac{\mu}{r_2^2-b^2}}=\frac{z_2'}{\frac{\nu}{r_2^2-c^2}}=L_2\ (\text{say}):$$

remembering that

$$a^2x_1'^2+b^2y_1'^2+c^2z_1'^2=1,$$

$$\text{and}\quad a^2x_2'^2+b^2y_2'^2+c^2z_2'^2=1,$$

since the points $R_1\ R_2$ are on the indicatrix.

). *The points $R_1 R_2$, corresponding to the rays Or_1 Or_2 transmissible the line $\lambda\mu\nu$, are in a plane conjugate to that line.*

nce the normals of the indicatrix at $R_1 R_2$ are both perpendicular to ine $\lambda \mu \nu$ we have

$$\lambda a^2 x_1' + \mu b^2 y_1' + \nu c^2 z_1' = 0$$
$$\lambda a^2 x_2' + \mu b^2 y_2' + \nu c^2 z_2' = 0.$$

nce the points $R_1 R_2$ are in the plane

$$\lambda a^2 x + \mu b^2 y + \nu c^2 z = 0.$$

is is the equation of a plane passing through the centre of the indicatrix parallel to the planes which touch the indicatrix at either of the points e the line $\lambda \mu \nu$ intersects it.

is also obvious geometrically that the tangent planes at $R_1 R_2$ are of them parallel to Or, and that Or is therefore parallel to their line tersection; Or is thus conjugate to the plane containing the points $_1$, R_2.

d it is geometrically evident that at *all* points of the section made by onjugate plane the tangent planes to the indicatrix are parallel to, and fore their normals perpendicular to, the conjugate line $\lambda \mu \nu$: the is $R_1 R_2$ are those of the section for which the normals of the indicatrix ot only perpendicular to the line $\lambda \mu \nu$, but intersect it.

.. *Given the direction of transmission, to find the positions of the cording points $R_1 R_2$ in the conjugate plane.*

om the last Article, it follows that the tangent planes to the indicaat its intersection with the conjugate plane form a tangent cylinder, ng its axis parallel to the direction of transmission. Let UKV be the e of contact of the cylinder and indicatrix (Fig. 8): $R_1 R_2$ are some- e on the curve UKV.

a line is only perpendicular to its conjugate plane when it coincides an axis of the indicatrix, Or, the axis of the cylinder, is in general ue to UKV, the conjugate plane.

t $U'K'V'$, $U''K''V''$ be sections of the cylinder by two planes perlicular to its axis; they are in general ellipses: let $K''KK'$ be any on the cylinder parallel to the axis, and $K''L''$, KL, $K'L'$, be the normals e cylinder at the points K'', K, K', respectively; they are evidently llel to each other.

t KL is also the normal of the indicatrix at K, for the cylinder and atrix are tangent to each other at that point: also $K'L'$ lies in the e $U'K'V'$, since that plane is perpendicular to the axis of the cylinder: e $K'L'$ is a normal of the ellipse $U'K'V'$ at the point K'.

The line KL will thus only intersect the axis of the cylinder when is an axis of the section $U'K'V'$.

Hence the points $R_1\ R_2\ \overline{R}_1\ \overline{R}_2$ are the four positions of K, on the c UKV, for which the normal of the indicatrix intersects the axis of cylinder: and these four positions are projections, by lines parall the axis of the cylinder, of the extremities of the axes of its "base;" base being taken as perpendicular to the axis of the cylinder.

In other words, the points $R_1 R_2$ and the normals $R_1N_1\ R_2N_2$ lie i *planes of symmetry of that tangent cylinder* of the indicatrix which ha axis in the common direction of transmission of the rays.

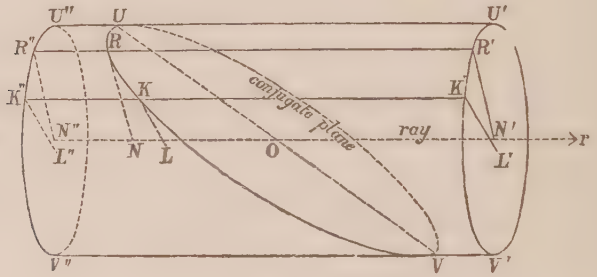

FIG. 8.

12. *The planes of polarisation of the two rays Or_1 Or_2 transmi along the same line are perpendicular to each other.*

By the last Article, the normals $R_1N_1\ R_2N_2$ are in the planes of metry of the tangent cylinder and at right angles to its axis: hence planes of polarisation, to which the two lines are perpendicular, are t selves perpendicular to each other.

The following analytical proof is interesting by reason of the eli tions:—

The normals of the planes of polarisation being normals of the indi at $R_1 R_2$, their direction-cosines are

$$\frac{a^2x_1'}{r_1},\ \frac{b^2y_1'}{r_1},\ \frac{c^2z_1'}{r_1},$$

$$\frac{a^2x_2'}{r_2},\ \frac{b^2y_2'}{r_2},\ \frac{c^2z_2'}{r_2},$$ respectively (Art. 5);

hence, if ϕ be the angle between the planes of polarisation,

$$\cos\phi=\frac{1}{r_1r_2}(a^4x_1'x_2'+b^4y_1'y_2'+c^4z_1'z_2').$$

ituting the values of $x_1'y_1'z_1'$, $x_2'y_2'z_2'$, from Article 9, we have

$$= \frac{L_1L_2}{r_1r_2}\left\{\frac{a^4\lambda^2}{(r_1^2-a^2)(r_2^2-a^2)}+\frac{b^4\mu^2}{(r_1^2-b^2)(r_2^2-b^2)}+\frac{c^4\nu^2}{(r_1^2-c^2)(r_2^2-c^2)}\right\}.$$

w r_1^2 r_2^2 being the roots of the equation given in Article 8, we have

$$r_1^2+r_2^2=\frac{a^2(b^2+c^2)\lambda^2+b^2(c^2+a^2)\mu^2+c^2(a^2+b^2)\nu^2}{a^2\lambda^2+b^2\mu^2+c^2\nu^2}$$

$$r_1^2r_2^2=\frac{a^2b^2c^2}{a^2\lambda^2+b^2\mu^2+c^2\nu^2};$$

$$(r_1^2-a^2)(r_2^2-a^2)=r_1^2r_2^2-a^2(r_1^2+r_2^2)+a^4$$

$$=-\frac{a^2\lambda^2}{b^2-c^2}\frac{(a^2-b^2)(b^2-c^2)(c^2-a^2)}{a^2\lambda^2+b^2\mu^2+c^2\nu^2}.$$

arly $$(r_1^2-b^2)(r_2^2-b^2)=-\frac{b^2\mu^2}{c^2-a^2}\frac{(a^2-b^2)(b^2-c^2)(c^2-a^2)}{a^2\lambda^2+b^2\mu^2+c^2\nu^2},$$

$$(r_1^2-c^2)(r_2^2-c^2)=-\frac{c^2\nu^2}{a^2-b^2}\frac{(a^2-b^2)(b^2-c^2)(c^2-a^2)}{a^2\lambda^2+b^2\mu^2+c^2\nu^2}.$$

$$\frac{a^4\lambda^2}{(r_1^2-a^2)(r_2^2-a^2)}+\frac{b^4\mu^2}{(r_1^2-b^2)(r_2^2-b^2)}+\frac{c^4\nu^2}{(r_1^2-c^2)(r_2^2-c^2)}=0,$$

is a right angle.

If ρ be the point in which Or intersects the indicatrix, the lines OR_1, OR_2 form a triad of conjugate diameters of the indicatrix.

axes of the basal section $U'K'V'$ (Fig. 8) of the tangent cylinder indicatrix being conjugate to each other, it follows, from the pro- of parallel projection, that the projections of the axes on any n of the cylinder are conjugate diameters of the curve of section: the line $O\rho$ and the lines OR_1 OR_2 (which are the projections of asal axes on the conjugate plane of Or), form a triad of conjugate ters of the indicatrix, each being conjugate to the plane of the other

s may also be proved analytically, as follows:—

stituting the values of x_1' y_1' z_1', x_2' y_2' z_2', given in Art. 9, we find

$$a^2x_1'x_2'+b^2y_1'y_2'+c^2z_1'z_2'=$$

$$L_2\left\{\frac{a^2\lambda^2}{(r_1^2-a^2)(r_2^2-a^2)}+\frac{b^2\mu^2}{(r_1^2-b^2)(r_2^2-b^2)}+\frac{c^2\nu^2}{(r_1^2-c^2)(r_2^2-c^2)}\right\}$$

e quantity within the brackets is zero, as may be seen from the last e, or more directly by subtraction of the equations

$$\frac{a^2\lambda^2}{r_1^2-a^2}+\frac{b^2\mu^2}{r_1^2-b^2}+\frac{c^2\nu^2}{r_1^2-c^2}=0$$

$$\frac{a^2\lambda^2}{r_2^2-a^2}+\frac{b^2\mu^2}{r_2^2-b^2}+\frac{c^2\nu^2}{r_2^2-c^2}=0.$$

Hence $a^2x_1'x_2' + b^2y_1'y_2' + c^2z_1'z_2' = 0$; but this is the condition the lines $OR_1\ OR_2$ may be parallel to the tangent planes at R_2 and respectively: as those tangent planes are likewise both parallel to the Or, the three lines form a triad of conjugate diameters.

Corollary. The normal of the indicatrix at the point R_1, thoug intersects the line Or, only intersects the line OR_2 when N_1 coincides O (Fig. 14); hence it follows that if OR_2 be a line of ray-transmiss R_1 is *not* one of the corresponding points on the indicatrix: in gen the lines of the conjugate triad $O\rho\ OR_1\ OR_2$ are thus not interchange in character.

14. *Given $r_1\ r_2$, the velocities of two rays which are transmitted in same direction, to find $\lambda\ \mu\ \nu$ the direction-cosines of the line of transmis*

From Article 8,

$$\frac{\lambda^2}{\frac{1}{r_1^2}-\frac{1}{a^2}} + \frac{\mu^2}{\frac{1}{r_1^2}-\frac{1}{b^2}} + \frac{\nu^2}{\frac{1}{r_1^2}-\frac{1}{c^2}} = 0,$$

$$\frac{\lambda^2}{\frac{1}{r_2^2}-\frac{1}{a^2}} + \frac{\mu^2}{\frac{1}{r_2^2}-\frac{1}{b^2}} + \frac{\nu^2}{\frac{1}{r_2^2}-\frac{1}{c^2}} = 0.$$

Determining the ratios $\lambda^2 : \mu^2 : \nu^2$ from these equations, we find th

$$\frac{\lambda^2}{\left(\frac{1}{b^2}-\frac{1}{c^2}\right)\left(\frac{1}{r_1^2}-\frac{1}{a^2}\right)\left(\frac{1}{r_2^2}-\frac{1}{a^2}\right)}$$

is equal to the two corresponding symmetrical fractions.

Each of the fractions is equal to the fraction of which the nume is the sum of the three numerators, and the denominator the su the three denominators.

The sum of the numerators of the three fractions is unity: the su the denominators is

$$\frac{1}{r_1^2 r_2^2}\left[\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right]$$

$$-\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)\left[\frac{1}{a^2}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+\frac{1}{b^2}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+\frac{1}{c^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\right]$$

$$+\frac{1}{a^4}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+\frac{1}{b^4}\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+\frac{1}{c^4}\left(\frac{1}{a^2}-\frac{1}{b^2}\right).$$

The coefficients of $\frac{1}{r_1^2 r_2^2}$ and $\frac{1}{r_1^2} + \frac{1}{r_2^2}$ vanish, and the remaining term may be transformed into

$$-\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{1}{b^2}-\frac{1}{c^2}\right)\left(\frac{1}{c^2}-\frac{1}{a^2}\right).$$

Hence
$$\lambda^2=\frac{\left(\frac{1}{r_1^2}-\frac{1}{a^2}\right)\left(\frac{1}{r_2^2}-\frac{1}{a^2}\right)}{\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\left(\frac{1}{c^2}-\frac{1}{a^2}\right)},$$

$$\mu^2=\frac{\left(\frac{1}{r_1^2}-\frac{1}{b^2}\right)\left(\frac{1}{r_2^2}-\frac{1}{b^2}\right)}{\left(\frac{1}{c^2}-\frac{1}{b^2}\right)\left(\frac{1}{a^2}-\frac{1}{b^2}\right)},$$

$$\nu^2=\frac{\left(\frac{1}{r_1^2}-\frac{1}{c^2}\right)\left(\frac{1}{r_2^2}-\frac{1}{c^2}\right)}{\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\left(\frac{1}{b^2}-\frac{1}{c^2}\right)}.$$

15. *Given the direction of a line of transmission, to find the velocities of the corresponding rays.*

If RR' (Fig. 8) be parallel to the axis of the cylinder, and R' be the extremity of an axis of the base, it follows from Article **11** that R is a point on the indicatrix corresponding to a ray transmissible along the axis of the cylinder. The corresponding velocity being $\frac{1}{RN} = \frac{1}{R'N'}$, it is seen that the velocities of transmission are inversely proportional to the axes of the base of the cylinder. An analytical solution is given in Art. 8.

16. The optic bi-radials (*secondary optic axes*).

From Article **15** it follows that if the two velocities of transmission in a given direction are equal, the corresponding tangent cylinder has a circular base. But at *every* point K' (Fig. 8) on the edge of the base of such cylinder, the normal of the basal section and therefore of the cylinder, and consequently also the normal of the cylinder and therefore of the indicatrix at every corresponding point K of the section conjugate to the base, intersect the axis of the cylinder perpendicularly, and have the same length intercepted between the surface and the axis: hence *every* point on the conjugate section corresponds to a ray transmissible with the same velocity along the axis of the cylinder: the normals of the indicatrix at these points, and therefore the planes of polarisation, may have any azimuth whatever.

That in the plane AOC there are two directions, and only two, name those of the lines Os_1 Os_2 (Fig. 7c), for which the two velocities of tran mission are equal, has already been proved (Art. 3).

Along each of these lines Os_1 Os_2, rays can thus be transmitted having a azimuth of plane of polarisation whatever, and the velocity of transmissi is b for all of them: in the case of calcite and analogous crystals, such pr perties only belong to that single direction which is termed the optic ax By reason of this analogy, the directions Os_1 Os_2 have been likewi termed optic axes.

But not being perpendicular to the corresponding ray-fronts, they not possess all the characters which belong to the optic axis of a uniax crystal: from another pair of directions, of which the optical characte are also such as in the case of a uniaxal crystal only belong to the opt axis, they have been distinguished as *Secondary Optic Axes;* and by S William Hamilton as *Lines of Single Ray-Velocity*.[1]

In the case of a biaxal crystal, it is experimentally determined th none of the so-called optic axes, primary or secondary, have directio which pass permanently through the same lines of crystalline particle the lines of particles through which they pass differ with the colour of t light and the temperature of the crystal: hence the so-called optic ax have no material existence, and are in no proper sense of the word *axes the crystal*.

Where precision of thought and language is necessary, the lines m appropriately be termed *the Optic Bi-radials*, for they are directions which a line is doubly a radius vector of the ray-surface: the term *u radial* has already been assigned a distinct signification by Mac Cullagh

When the indicatrix is a spheroid at all temperatures of the crystal a for all colours of light, the bi-radial is found to be an axis of mo phological and physical symmetry, and an axis of revolution of the ra surface; it always passes through the same line of crystalline particle such a line may be regarded as a true axis of the crystal.

17. *There cannot be more than one pair of optic bi-radials.*

It has already been proved that Os_1 Os_2 are the only directions which the velocities of the rays transmissible along the same line, ly in a plane of symmetry of the indicatrix, are equal: it remains to pr that there are no other bi-radials in any direction whatever.

[1] *Trans. Roy. Irish Acad.*: 1837, vol. 17, p. 132.

[2] *Ibid.*: 1839, vol. 18, p. 40.

rom Article 8 it is seen that the velocity of transmission r is connected the values of $\lambda\ \mu\ \nu$ by the equation

$$\frac{a^2\lambda^2}{r^2-a^2}+\frac{b^2\mu^2}{r^2-b^2}+\frac{c^2\nu^2}{r^2-c^2}=0;$$

ace

$$^2\lambda^2(r^2-b^2)(r^2-c^2)+b^2\mu^2(r^2-c^2)(r^2-a^2)+c^2\nu^2(r^2-a^2)(r^2-b^2)=0.$$

e $a\ b\ c$ are in descending order of magnitude, the expression on the nand side of the last equation is positive, and therefore cannot be , if r has any value greater than a or less than c: hence no velocity ransmitted ray can be greater than a or less than c.

rther, if μ is distinct from zero, the above expression is necessarily itive when $r=b$; hence it changes sign and passes through a zero as r decreases from a to b, and again as r decreases from b to c. is distinct from zero, the two values of r^2 which satisfy the above ion are thus unequal.

nce the bi-radials can only lie in the plane AOC.

at in the plane AOC there are only two such lines may also be seen the fact that for any direction lying in this plane one velocity of nission is always b: when the two velocities are equal the second ity must also be b: hence if S is a point on the curve $AC\overline{A}\overline{C}$ such the perpendicular to the radius vector conjugate to OS is equal to the points S correspond to directions Os of single ray-velocity: there ur such points lying at the extremities of two diameters.

e directions Os may also be readily found from the above general ion: for all rays lying in the plane AOC, μ is zero, and the general on becomes

$$\frac{a^2\lambda^2}{r^2-a^2}+\frac{c^2\nu^2}{r^2-c^2}=0;$$

e, the rays in this plane for which the two velocities of transmission oth equal to b are given by the equation

$$\frac{a^2\lambda^2}{b^2-a^2}+\frac{c^2\nu^2}{b^2-c^2}=0,$$

h is identical with the equation given in Article 3.

4. *Equation of the planes conjugate to the optic bi-radials.*

e equation of a plane conjugate to a line $\lambda\ \mu\ \nu$ is

$$a^2x\lambda+b^2y\mu+c^2z\nu=0.$$

he bi-radials, $\mu=0$ and

$$\frac{a^2\lambda^2}{b^2-a^2}+\frac{c^2\nu^2}{b^2-c^2}=0.$$

Hence their conjugate planes are given by the equation

$$\frac{x^2}{c^2(b^2-c^2)} = \frac{z^2}{a^2(a^2-b^2)}.$$

19. *The direction of a line Or being defined by its inclinations* σ_1 *to the bi-radials* Os_1 Os_2, *to find the planes of polarisation of the tw rays which can be transmitted along it.*

Let $[s_1]$, $[s_2]$, $[r]$ (Fig. 9) be the sections of the indicatrix which ar conjugate to the lines Os_1, Os_2, Or respectively, and let D_1 D_2 be tw adjacent points of intersection of $[r]$ with $[s_1]$ and $[s_2]$.

D_1 being common to the sections $[r]$ and $[s_1]$, the tangent plane of th indicatrix at D_1 is parallel to both Or and Os_1, and therefore to the plan Ors_1 containing them. Similarly the tangent plane of the indicatrix a D_2 is parallel to the plane Ors_2.

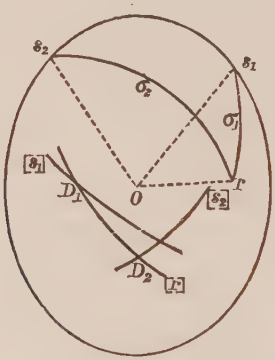

FIG. 9.

Also all planes tangent to the indicatrix at points on the sections $[s$ and $[s_2]$ are equidistant from the origin (Art. **16**): hence the tange planes of the indicatrix at D_1 and D_2 are equidistant from the line Or, an are therefore equally inclined to the planes of polarisation, for the latt are the planes of symmetry of the elliptic cylinder which touches th ellipsoid in the section $[r]$ (Art. **12.**).

Hence the planes of polarisation of the two rays transmissible in th direction Or are the internal and external bisectors of the angle betwe the planes Ors_1 Ors_2.

This is the first of the empirical laws of Biot (page 801).

20. *The direction of a line Or being defined by its inclinations σ_1 σ_2 the bi-radials Os_1 Os_2, to find the velocities of the two rays which can transmitted along it.*

If r_1 r_2 be the respective velocities, it follows from the quadratic equation Art. 8 that

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}=\left(\frac{1}{b^2}+\frac{1}{c^2}\right)\lambda^2+\left(\frac{1}{c^2}+\frac{1}{a^2}\right)\mu^2+\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\nu^2$$

$$\frac{1}{r_1^2 r_2^2}=\frac{\lambda^2}{b^2c^2}+\frac{\mu^2}{c^2a^2}+\frac{\nu^2}{a^2b^2}$$

where λ μ ν are the direction-cosines of Or.

It is necessary to express λ μ ν in terms of the angles σ_1 σ_2, and to substitute their values in the above expressions.

Let l 0 n, $\bar{l}$ 0 n be the direction-cosines of Os_1 Os_2 respectively, then

$$\cos\sigma_1 = l\lambda+n\nu, \qquad \cos\sigma_2 = -l\lambda+n\nu;$$

$$2l\lambda = \cos\sigma_1 - \cos\sigma_2, \qquad 2n\nu = \cos\sigma_1 + \cos\sigma_2.$$

Also (from Art. 3)

$$l^2=\frac{\frac{1}{a^2}-\frac{1}{b^2}}{\frac{1}{a^2}-\frac{1}{c^2}}, \qquad n^2=\frac{\frac{1}{b^2}-\frac{1}{c^2}}{\frac{1}{a^2}-\frac{1}{c^2}}.$$

Substituting $1-\lambda^2-\nu^2$ for μ^2 in the expression for $\frac{1}{r_1^2}+\frac{1}{r_2^2}$, we get

$$\begin{aligned}\frac{1}{r_1^2}+\frac{1}{r_2^2}&=\frac{1}{a^2}+\frac{1}{c^2}-\lambda^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\nu^2\left(\frac{1}{b^2}-\frac{1}{c^2}\right)\\&=\frac{1}{a^2}+\frac{1}{c^2}-\left(\frac{1}{a^2}-\frac{1}{c^2}\right)(l^2\lambda^2-n^2\nu^2)\\&=\frac{1}{a^2}+\frac{1}{c^2}+\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\cos\sigma_1\cos\sigma_2.\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{1}{r_1^2r_2^2}&=\frac{1}{a^2c^2}-\frac{\lambda^2}{c^2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\frac{\nu^2}{a^2}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)\\&=\frac{1}{a^2c^2}-\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\left(\frac{l^2\lambda^2}{c^2}-\frac{n^2\nu^2}{a^2}\right)\\&=\frac{1}{a^2c^2}-\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\left[\frac{(\cos\sigma_1-\cos\sigma_2)^2}{4c^2}-\frac{(\cos\sigma_1+\cos\sigma_2)^2}{4a^2}\right]\end{aligned}$$

$$\frac{4}{r_1^2r_2^2}=\frac{4}{a^2c^2}+\left(\frac{1}{a^2}-\frac{1}{c^2}\right)^2(\cos^2\sigma_1+\cos^2\sigma_2)+2\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\cos\sigma_1\cos\sigma_2.$$

Hence

$$\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)^2=\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)^2-\frac{4}{r_1^2r_2^2}$$

$$=\left(\frac{1}{a^2}-\frac{1}{c^2}\right)^2\sin^2\sigma_1\sin^2\sigma_2,$$

$$\text{or}\quad \frac{1}{r_1^2}-\frac{1}{r_2^2}=\pm\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\sin\sigma_1\sin\sigma_2.$$

If r_1 be the larger velocity, we thus have

$$\frac{1}{r_1^2}-\frac{1}{r_2^2}=\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\sin\sigma_1\sin\sigma_2.$$

This is the second of the empirical laws of Biot (page 301).
From the above we find that

$$\frac{2}{r_1^2}=\frac{1}{a^2}+\frac{1}{c^2}+\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\cos(\sigma_1-\sigma_2)$$

$$\frac{2}{r_2^2}=\frac{1}{a^2}+\frac{1}{c^2}+\left(\frac{1}{a^2}-\frac{1}{c^2}\right)\cos(\sigma_1+\sigma_2):$$

whence

$$\frac{1}{r_1^2}=\frac{1}{a^2}\cos^2\frac{\sigma_1-\sigma_2}{2}+\frac{1}{c^2}\sin^2\frac{\sigma_1-\sigma_2}{2}$$

$$\frac{1}{r_2^2}=\frac{1}{a^2}\cos^2\frac{\sigma_1+\sigma_2}{2}+\frac{1}{c^2}\sin^2\frac{\sigma_1+\sigma_2}{2}.$$

21. *The ray-front corresponding to the ray Or is perpendicular to transverse plane RNOr, and intersects that plane in a line parallel to O*

Geometrical Proof.

(*a*.) As usual, let R (Fig. 10) be the point of the indicatrix corr ponding to the ray Or, and RN be the normal of the indicatrix at R: a let $RDEN$ be a plane perpendicular to the plane $RNOr$; it will inters the indicatrix in an ellipse. Let D be a point on this ellipse distant fr R by an arc which is a small quantity of the first order: to this order small quantities the normal of the ellipse at D is the normal of indicatrix at that point: if OE be drawn to intersect DE perpendicular the line EN is, to the first order of small quantities, perpendicular to plane $RNOr$ and parallel to DR, and DE is equal to RN. Hence, if be the ray corresponding to the point D of the indicatrix, the plane is perpendicular to the plane $RNOr$, and $Od=\frac{1}{DE}$ by construction $=$ $=Or$. Hence the line rd is also perpendicular to the plane $RNOr$.

But R and D being adjacent points on the indicatrix, r and d are adjacent nts on the ray-surface, and the line rd is thus tangent to the ray-face at r.

The tangent plane to the ray-surface at r is the plane of the ray-front rresponding to the ray Or: it must pass through all lines tangent to ray-surface at r, and thus through the line rd, and be perpendi-ar therefore to the plane $RNOr$.

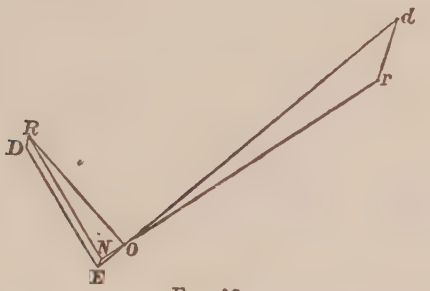

Fig. 10.

(*b*.) The plane $RNOr$ (Fig. 11) also will intersect the indicatrix in ellipse: let G be a point on this ellipse distant from R by an arc ich is a small quantity of the first order: to this order of small quanti-s GH, the normal of the ellipse at G, is also the normal of the indicatrix that point. Hence if gOH be perpendicular to GH and $Og=\frac{1}{GH}$, is the direction of the ray corresponding to the point G, and g is a int on the ray-surface: in the same way as before it follows that rg is tangent line of the ray-surface, and is thus the intersection of the ngent plane at r with the plane $RNOr$.

Let Or Og intersect the ellipse in the points R' G' respectively: then R' and OG' are respectively conjugate to OR and OG, being perpen-ular to RN and GH, and therefore parallel to the tangents at R and G: e area of a parallelogram of which the adjacent sides are conjugate lii vectores is constant; hence

$$RN\cdot OR'=GH\cdot OG'.$$

so, by construction,

$$RN=\frac{1}{Or},\quad GH=\frac{1}{Og};$$

hence

$$Or:OR'=Og:OG',$$

and the line rg is therefore parallel to the tangent of the ellipse at R and consequently to the line OR which is conjugate to OR'.

Hence the ray-front corresponding to the ray Or intersects the transverse plane $RNOr$ perpendicularly in a line parallel to OR.

The diametral line Of, perpendicular to OR and lying in the plane RNO is therefore normal to the ray-front corresponding to the ray Or (Fig. 11).

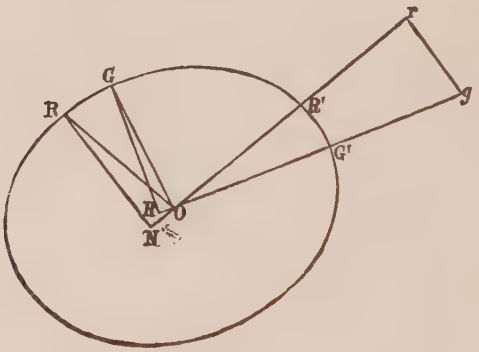

Fig. 11.

Analytical Proof.

The following is interesting to the mathematical student, by reason the eliminations:—

From Article 7 we have

$$\frac{\frac{x}{r^2-a^2}}{x'} = \frac{\frac{y}{r^2-b^2}}{y'} = \frac{\frac{z}{r^2-c^2}}{z'} = \frac{1}{A} \text{ (say)};$$

hence $$x' = \frac{Ax}{r^2-a^2},\quad y' = \frac{Ay}{r^2-b^2},\quad z' = \frac{Az}{r^2-c^2}. \qquad (1).$$

Remembering that

$$\frac{a^2}{r^2-a^2} + \frac{y^2}{r^2-b^2} + \frac{z^2}{r^2-c^2} = 1 \text{ (Art. 7)},$$

we have $$xx' + yy' + zz' = A. \qquad (2).$$

Also $$a^2xx' + b^2yy' + c^2zz' = 0 \text{ (Art. 7)}. \qquad (3).$$

It is thus required to determine the tangent plane at a point $x\,y\,z$ of ray-surface in terms of the co-ordinates $x'y'z'$, which are connected by above equations and also by the relation

$$a^2x'^2 + b^2y'^2 + c^2z'^2 = 1. \qquad (4).$$

forming the differential of each of the equations (1), we have

$$\left.\begin{aligned}(r^2-a^2)\delta x' + 2rx'\delta r &= A\delta x + x\delta A\\ (r^2-b^2)\delta y' + 2ry'\delta r &= A\delta y + y\delta A\\ (r^2-c^2)\delta z' + 2rz'\delta r &= A\delta z + z\delta A\end{aligned}\right\} \qquad (5).$$

Multiply these equations by a^2x' b^2y' c^2z' respectively and add: the quantity δA is thus eliminated, for its coefficient $a^2xx' + b^2yy' + c^2zz'$ vanishes by relation (3); we then have

$$(r^2-a^2)a^2x'\delta x' + (r^2-b^2)b^2y'\delta y' + (r^2-c^2)c^2z'\delta z' + 2r\delta r = A\,(a^2x'\delta x + b^2y'\delta y + c^2z'\delta z).$$

Remembering that $a^2x'\delta x' + b^2y'\delta y' + c^2z'\delta z' = 0$, owing to relation (4), we have

$$\cdots - (a^4x'\delta x' + b^4y'\delta y' + c^4z'\delta z') + 2r\delta r = A(a^2x'\delta x + b^2y'\delta y + c^2z'\delta z).$$

But by Article 4

$$a^4x'^2 + b^4y'^2 + c^4z'^2 = r^2;$$

hence

$$a^4x'\delta x' + b^4y'\delta y' + c^4z'\delta z' = r\delta r.$$

Substituting this value in the preceding equation, we have

$$r\delta r = A(a^2x'\delta x + b^2y'\delta y + c^2z'\delta z)$$

$$(x - Aa^2x')\delta x + (y - Ab^2y')\delta y + (z - Ac^2z')\delta z = 0.$$

If l m n be the direction-cosines of Of, the perpendicular to the tangent plane of the ray-surface at x y z, we must have

$$l\delta x + m\delta y + n\delta z = 0, \text{ whence}$$

$$\frac{l}{x - Aa^2x'} = \frac{m}{y - Ab^2y'} = \frac{n}{z - Ac^2z'}; \qquad (6).$$

(a.) Each of these quantities is equal to

$$\frac{lx' + my' + nz'}{xx' - Aa^2x'^2 + yy' - Ab^2y'^2 + zz' - Ac^2z'^2}$$

or

$$\frac{lx' + my' + zz'}{xx' + yy' + zz' - A}.$$

The denominator of this expression is zero, by relation (2); hence the numerator $lx' + my' + nz'$ is also zero, for the three equivalent fractions are either all of them indeterminate, and are none of them infinite.

From the relation

$$lx' + my' + nz' = 0, \qquad (7).$$

it follows that Of is perpendicular to OR.

(b.) Also, multiplying both numerator and denominator of each of the fractions (6) by $y'z'(b^2-c^2)$, $z'x'(c^2-a^2)$, $x'y'(a^2-b^2)$ respectively, we find that each of them is equal to

$$\frac{ly'z'\,(b^2-c^2) + mz'x'\,(c^2-a^2) + nx'y'\,(a^2-b^2)}{y'z'\,(b^2-c^2)\,(x - Aa^2x') + z'x'\,(c^2-a^2)\,(y - Ab^2y') + x'y'\,(a^2-b^2)\,(z - Ac^2z')}.$$

On expanding the denominator, it will be found that the terms involvi
A mutually destroy each other, owing to the identity

$$a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)=0:$$

the denominator thus reduces to

$$xy'z'(b^2-c^2)+yz'x'(c^2-a^2)+zx'y'(a^2-b^2)\ ;\ \text{or}$$

$$\frac{x'y'z'}{A}\left\{(r^2-a^2)(b^2-c^2)+(r^2-b^2)(c^2-a^2)+(r^2-c^2)(a^2-b^2)\right\}$$

owing to the equations (1).

When multiplied out, this term is likewise found to be zero.

Hence the numerator of the above expression is also zero, and we ha
he relation

$$ly'z'(b^2-c^2)+mz'x'(c^2-a^2)+nx'y'(a^2-b^2)=0. \quad (8).$$

But this is the condition (Art. 6) that the line Of may lie in the pla
RON: hence the front-normal Of lies in the plane $RNOr$ and is perp
dicular to OR.

Corollary 1. The inclination of a ray Or to its front-normal Of is
same as the inclination of the normal RN to the radius vector RO
the corresponding point R of the indicatrix (Fig. 12).

Corollary 2. If a ray coincides with the central normal to its r
front, its direction is perpendicular to an axis of the indicatrix.

Corollary 3. If the ray Or lies in one of the symmetral planes
the indicatrix, the intersection of the corresponding ray-front with
symmetral plane is parallel to the line OR, which is conjugate to
line Or. But if P is any point on an ellipse and $Q\ \overline{Q}$ are the extremit
of a diameter, the lines $PQ\ P\overline{Q}$ are parallel to a pair of conjugate d
meters. Hence, if $P\ Q\ \overline{Q}$ lie in a symmetral plane of the indicat
and PQ represents the direction of a ray, the corresponding ray-front
a plane perpendicular to the symmetral plane, and intersects the lat
in a line parallel to $P\overline{Q}$.

22. *For the ray Or, the plane of polarisation is perpendicular to
plane containing Or and Of the normal of the corresponding ray-front.*

This follows at once from the last Article, for RN, the normal
the plane of polarisation of the ray, lies in the plane $RNOr$ which
been shown to contain the line Ot.

In other words, the plane Orf is the tranverse plane for the ray Or.

23. *For the ray Or, corresponding to the point R, the resol
velocity along the normal to the ray-front is measured by the inverse of O*

If Of (Fig. 12) be perpendicular to OR and in the plane $RNOr$,

angle rfO be a right angle, then, by Article **21**, f is the foot of perpendicular drawn from O to the tangent plane of the ray-surface r, and Of is the resolved velocity of the ray Or along the normal to corresponding front.

But Or Of are by construction perpendicular respectively to RN and : : hence the triangles rfO ONR are similar and $\frac{RN}{OR}=\frac{Of}{Or}$.

Also, by construction, $RN=\frac{1}{Or}$; hence $Of=\frac{1}{OR}$.

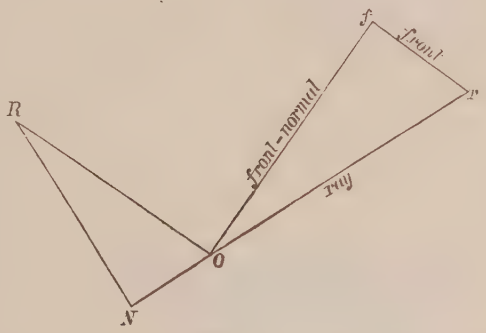

Fig. 12.

24. *The line OR is always a normal of the curve in which the indicatrix is intersected by a central plane parallel to that ray-front which corresponds to the ray Or: in the general case, OR is an axis of the curve.*

RN (Fig. 12) being the normal of the indicatrix at R, any line perpendicular to RN and to the plane $RNOr$ is tangent both to the indicatrix, and to the section of the indicatrix made by any plane which is perpendicular to the plane $RNOr$ at the point R; it is thus tangent to the elliptic section made by that plane of the series which passes through O. This section OR is a central radius vector: hence the tangent at R to the section is at right angles to a central radius vector.

The section being in general an ellipse, R is in such case the extremity of an axis of the section.

Hence it is seen that the ray-surface is the envelope of planes which are distant from a parallel central section of the indicatrix by the inverse lengths of the semi-axes of the latter curve: which is Fresnel's geometrical construction of the surface.

Conversely,

25. *If OR is a central normal (and therefore in general an axis) the curve in which the indicatrix is intersected by a plane parallel to a giv direction of ray-front, the plane through OR normal to the direction of t ray-front contains the ray Or, which corresponds to the point R, and al the line RN, which is the normal of the plane of polarisation of the ray.*

The radius vector OR (Fig. 12) being a central normal of the cur of intersection, a line perpendicular to OR and lying in the plane paral to the ray-front, is a tangent to the curve of intersection at R: hen RN the normal of the indicatrix at R must lie somewhere or other in t plane ROf perpendicular to this line.

And the ray Or must lie in the same plane.

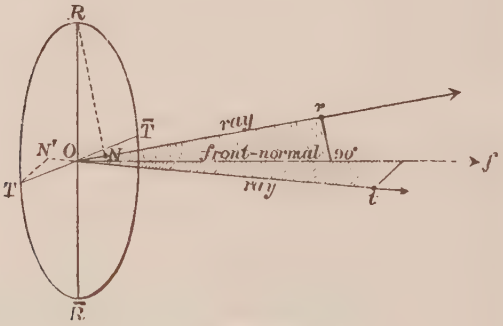

FIG. 13.

26. The two rays corresponding to a given directi of front-normal.

Hence if only the *direction of a ray-front* be given, there are in gen two corresponding *positions* of the ray-front, or, in other words, of tang planes to the ray-surface: and for each there is a corresponding ray (Fig. The rays lie each of them in a plane containing the central normal Of one of the axes OR OT of the section of the indicatrix by a plane para to the ray-front; they are thus in two perpendicular planes which in sect in the line Of: the corresponding velocities resolved along the gi front-normal are measured by $\frac{1}{OR}$ and $\frac{1}{OT}$ respectively: the normal of plane of polarisation is parallel to RN for the ray Or and to TN' for ray Ot. The directions of vibration at points of the respective rays,

ng to the latest version of the elastic theory, are thus indicated by
nading in Figure 13.

may be remarked that the planes of polarisation of the rays Or Oi,
h perpendicular to the normals RN TN', are not perpendicular to
other; for it is the lines RO TO, not the lines RN TN', which are
ht angles: it is easily seen that the cosine of the angle between the
s is equal to sin fOr sin fOt. Hence only the transverse planes, not
lanes of polarisation of the two rays, are perpendicular to each other.

'. **The two front-normals corresponding to a given ction of ray.**

milarly, if only the *direction of a ray* be given, there are in general
corresponding *positions and directions* of the ray-front, and two
sponding rays (Fig. 14). The rays lie each of them in a plane

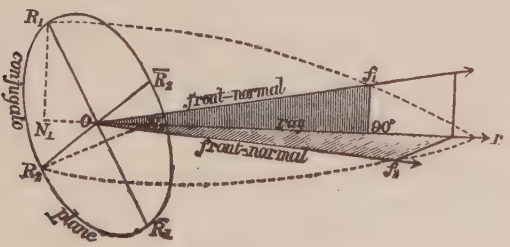

FIG. 14.

ining the ray-direction and one of the lines OR_1 OR_2: they are thus
perpendicular planes which intersect in the line Or: the corres-
ing velocities are measured by $\frac{1}{R_1N_1}$ $\frac{1}{R_2N_2}$, respectively: the normal
e plane of polarisation is parallel to R_1N_1 for the ray Or_1, and to
for the ray Or_2. The directions of vibration at points of the re-
ive rays, according to the latest version of the elastic theory, are
indicated by the shading in Figure 14.

3. *Given the co-ordinates $x'y'z'$ of R, to find θ, the angle between
orresponding ray Or and its front-normal Of.*

Of being perpendicular to RN RO respectively (Fig. 12),

$$\cos\theta = \cos rOf = \cos NRO = \frac{RN}{RO} = \frac{1}{rr'}.$$

hence $\tan^2\theta = r^2r'^2 - 1$

$= (a^4x'^2 + b^4y'^2 + c^4z'^2)(x'^2 + y'^2 + z'^2) - (a^2x'^2 + b^2y'^2 + c^2z'^2)^2$

$= (a^2 - b^2)^2x'^2y'^2 + (b^2 - c^2)^2y'^2z'^2 + (c^2 - a^2)^2z'^2x'^2.$

29. *Given the direction-cosines $\lambda\ \mu\ \nu$ of a line of transmission, to f θ, the angle between the corresponding ray Or and its front-normal Of.*

Find $r_1^2\ r_2^2$ (Art. 8), and then $x_1'y_1'z_1'\ x_2'y_2'z_2'$, the co-ordinates of points R_1R_2 (Art. 9); also $r_1'^2 = x_1'^2 + y_1'^2 + z_1'^2$ and $r_2'^2 = x_2'^2 + y_2'^2 + z$ finally we have $\sec\theta_1 = r_1r_1'$; $\sec\theta_2 = r_2r_2'$.

30. *If a ray lies in a given axial plane of the indicatrix, to find direction for which the inclination θ to the front-normal is a maximum.*

First proof.

Let the given axial plane be AOC. Each direction of transmiss lying in this plane corresponds to two points R_1R_2 on the indicatrix : of these R_2 always coincides with B, and the corresponding ray coinci with its front normal; the other R_1 is in the plane AOC, and the corr ponding ray coincides with its front-normal only when R_1 is at $A\bar{A}\ C$ o

If $x'\ 0\ z'$ be the co-ordinates of R_1, $\tan\theta = (c^2 - a^2)\ z'x'$ (Art. 28).

Hence, writing $c^2z'^2 = 1 - a^2x'^2$, we have

$$\tan^2\theta = \left(\frac{c}{a} - \frac{a}{c}\right)^2 (1 - a^2x'^2)\ a^2x'^2 = \left(\frac{c}{a} - \frac{a}{c}\right)^2 \left\{\tfrac{1}{4} - (a^2x'^2 - \tfrac{1}{2})^2\right\}$$

For a maximum value of θ, $a^2x'^2 = \frac{1}{2} = c^2z'^2$.

Or being parallel to the tangent of the indicatrix at R_1,

$$\tan rOA = -\frac{a^2x'}{c^2z'} = \pm\frac{a}{c}$$

or Or is parallel to AC or $\bar{A}C$, when the inclination to front-normal is a maximum.

Second proof.

From Article 21, Corollary 3, if P is a point on the indicatrix ly in the symmetral plane AOC, and PA represents the direction of a belonging to the elliptic section, the corresponding ray-front is perp dicular to the plane AOC and intersects it in a line parallel to $P\bar{A}$; he the angle $AP\bar{A}$ is the angle of inclination of the ray to its front : interior angle is a minimum when P coincides with C or $\bar{C}$,

. *If a ray lies in a given axial plane of the indicatrix, to find the aum inclination of the ray to its front-normal.*

rst proof.

iing AOC for the given axial plane, as in the preceding Article, we when $a^2x'^2=\frac{1}{2}$,

$$\tan\theta=\pm\tfrac{1}{2}\left(\frac{c}{a}-\frac{a}{c}\right),$$

ot $\theta=\tan 2\,ACO$, or $\theta=\frac{\pi}{2}-2\,ACO$.

nce the maximum or minimum angle which a ray lying in the axial AOC can make with its front is given by the angle $AC\overline{A}$.

ond proof.

s result is also manifest from the fact that when the ray Or is paral- $\overline{A}C$, the conjugate diameter OR_1, and therefore also the ray-front, is el to CA; as in the second proof given in Art. 30.

. *Given the co-ordinates $x'y'z'$ of R, to find the direction-cosines $l\,m\,n$ normal Of to the corresponding ray-front.*

any line $l\,m\,n$ in the plane $RNOr$, as already proved in Article 6, ve the equation

$$lx'(b^2-c^2)+mz'x'(c^2-a^2)+nx'y'(a^2-b^2)=0.$$

he line $l\,m\,n$ is likewise perpendicular to OR, of which the direc- sines are $\frac{x'}{r'}, \frac{y'}{r'}, \frac{z'}{r'}$, we have also

$$lx'+my'+nz'=0.$$

n these equations the ratios $l:m:n$ are found to be:—

$$\frac{l}{D}=\frac{m}{E}=\frac{n}{F}$$

where

$$D=x'z'^2(c^2-a^2)-x'y'^2(a^2-b^2)=x'(1-a^2r'^2),$$
$$E=y'x'^2(a^2-b^2)-y'z'^2(b^2-c^2)=y'(1-b^2r'^2),$$
$$F=z'y'^2(b^2-c^2)-z'x'^2(c^2-a^2)=z'(1-c^2r'^2);$$

whence

$$\frac{l}{x'(1-a^2r'^2)}=\frac{m}{y'(1-b^2r'^2)}=\frac{n}{z'(1-c^2r'^2)},$$

where $r'^2=OR^2=x'^2+y'^2+z'^2$.

se equations determine the direction-cosines $l\,m\,n$ of the normal the ray-front corresponding to the point $x'y'z'$ or R.

33. *Given the direction-cosines l m n of the front-normal Of, to f*
f_1 f_2 the respective velocities of the two corresponding ray-fronts resol
normally to them.

In Article **23** it was shown that the velocity of the ray-front of the
Or resolved normally to the front is $\frac{1}{Oli}$: denoting the resolved velocity
f and substituting $f=\frac{1}{Oli}=\frac{1}{r'}$ in the equations of the last Article,
have

$$\frac{l}{\overline{f^2-a^2}}=\frac{m}{\overline{f^2-b^2}}=\frac{n}{\overline{f^2-c^2}}.$$

Each of these fractions is equal to

$$\frac{l\overline{f^2-a^2}+m\overline{f^2-b^2}+n\overline{f^2-c^2}}{lx'+my'+nz'}.$$

But the denominator of the last expression has been proved to
zero (Art. **32**); hence the numerator is also zero, for the fracti
equivalent to the expression are never all of them indeterminate and
never infinite; we thus have

$$\frac{l^2}{f^2-a^2}+\frac{m^2}{f^2-b^2}+\frac{n^2}{f^2-c^2}=0.$$

This is a quadratic equation in f^2, and its two roots f_1^2 f_2^2 are the reso
velocities required.

Multiplied out it takes the form

$$f^4-f^2\{(b^2+c^2)l^2+(c^2+a^2)m^2+(a^2+b^2)n^2\}+b^2c^2l^2+c^2a^2m^2+a^2b^2n^2=0$$

34. *Given f_1 f_2, the velocities of normal-transmission of two*
fronts having a common direction of normal Of_1 f_2, to find l m n the di
tion-cosines of the latter.

From Art. **33**,

$$\frac{l^2}{f_1^2-a^2}+\frac{m^2}{f_1^2-b^2}+\frac{n^2}{f_1^2-c^2}=0$$

$$\frac{l^2}{f_2^2-a^2}+\frac{m^2}{f_2^2-b^2}+\frac{n^2}{f_2^2-c^2}=0.$$

Determining the ratios $l^2:m^2:n^2$ from these equations, we find that

$$\frac{l^2}{(b^2-c^2)(f_1^2-a^2)(f_2^2-a^2)}=\frac{m^2}{(c^2-a^2)(f_1^2-b^2)(f_2^2-b^2)}=\frac{n^2}{(a^2-b^2)(f_1^2-c^2)(f_2^2-}$$

ıe sum of the numerators of the fractions is unity: the sum of the
minators is

$$f_1^2 f_2^2 [(b^2-c^2)+(c^2-a^2)+(a^2-b^2)]$$
$$-(f_1^2+f_2^2)[a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)]$$
$$+a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2).$$

ıe coefficients of $f_1^2 f_2^2$ and $f_1^2+f_2^2$ vanish, and the remaining term
t be transformed into

$$-(a^2-b^2)(b^2-c^2)(c^2-a^2).$$

Hence
$$l^2=\frac{(f_1^2-a^2)(f_2^2-a^2)}{(b^2-a^2)(c^2-a^2)},$$
$$m^2=\frac{(f_1^2-b^2)(f_2^2-b^2)}{(c^2-b^2)(a^2-b^2)},$$
$$n^2=\frac{(f_1^2-c^2)(f_2^2-c^2)}{(a^2-c^2)(b^2-c^2)}.$$

5. *Given the direction-cosines l m n of the normals Of_1 Of_2 of two fronts having the same direction, to find the co-ordinates $x_1'y_1'z_1'$ $x_2'y_2'z_2'$ of the corresponding points R T on the indicatrix.*

ıe values $f_1^2 f_2^2$ having been found by the equation of Article 33, co-ordinates $x_1'\ y_1'\ z_1'$, $x_2'\ y_2'\ z_2'$ of the points R and T respectively termined by the following equations, also from Art. 33:—

$$\frac{x_1'}{l}=\frac{y_1'}{m}=\frac{z_1'}{n};$$
$$\frac{x_2'}{l}=\frac{y_2'}{m}=\frac{z_2'}{n};$$

mbering, also, that the co-ordinates of each point must satisfy the ition of the indicatrix.

will be observed that the above equations are identical in form with e given in Art. 9: in the one case the direction-cosines and velocities hose belonging to the rays, and in the other case are those belonging e front-normals.

3. *Given $f_1 f_2$, the velocities of normal-transmission of two ray-fronts ng a common direction of normal $Of_1 f_2$, to find the co-ordinates $x_1'y_1'z_1'$ $x_2'y_2'z_2'$ of the corresponding points R T on the indicatrix.*

rom Art. 35

$$\frac{x_1'}{l}=\frac{y_1'}{m}=\frac{z_1'}{n}=\frac{1}{A}\ (\text{say}),$$

The square of each of these fractions is equal to

$$\frac{x_1'^2+y_1'^2+z_1'^2}{\frac{l^2}{(f_1^2-a^2)^2}+\frac{m^2}{(f_1^2-b^2)^2}+\frac{n^2}{(f_1^2-c^2)^2}}.$$

Hence, remembering that $x_1'^2+y_1'^2+z_1'^2=OR^2=\frac{1}{f_1^2}$, we find o substituting the values of $l^2\ m^2\ n^2$ given in Art. 34

$$-\frac{A^2}{f_1^2}=\Sigma\frac{f_2^2-a^2}{(c^2-a^2)(a^2-b^2)(f_1^2-a^2)}$$

$$\text{or}\quad\frac{\Sigma\{(b^2-c^2)(f_1^2-b^2)(f_1^2-c^2)(f_2^2-a^2)\}}{(a^2-b^2)(b^2-c^2)(c^2-a^2)(f_1^2-a^2)(f_1^2-b^2)(f_1^2-c^2)}$$

On expansion of the numerator it will be seen that the terms involvin $f_1^4f_2^2\ f_1^2f_2^4\ f_1^4\ a^2b^2c^2$ all vanish, and that the coefficients of f_1^2 and f_2^2 ar equal but of opposite sign: the numerator then takes the form

$$(f_1^2-f_2^2)(a^2-b^2)(b^2-c^2)(c^2-a^2)$$

$$\text{Hence}\quad A^2=-\frac{f_1^2(f_1^2-f_2^2)}{(f_1^2-a^2)(f_1^2-b^2)(f_1^2-c^2)},$$

$$\text{and}\quad x_1'^2=\frac{l^2}{A^2(f_1^2-a^2)^2}=\frac{(f_2^2-a^2)(f_1^2-b^2)(f_1^2-c^2)}{f_1^2(f_1^2-f_2^2)(c^2-a^2)(a^2-b^2)}.$$

Corresponding symmetrical expressions give the values of $y_1'^2z_1'^2\ x_2'^2y_2'^2z_2'^2$

The above relation, with many others of this Section, was first give by Sylvester,[1] starting from the vibrational inferences of Fresnel.

37. *Given the direction-cosines $l\ m\ n$ of a front-normal Of, to fi those of the corresponding rays Or Ot.*

Find the co-ordinates of R and T by the method of Art. 35, and then th direction-cosines of the rays Or Ot by means of the equations in Article

38. *Given the direction-cosines of a ray Or, to find those of the co responding front-normals.*

Find the co-ordinates of R_1 and R_2 (Art. 9), and then the directio cosines of the corresponding front-normals by Article 32.

39. *The front-normal surface, or pedal of the ray-surface.*

It was shown in Art. 33 that if f be the velocity of transmission a ray-front resolved along its normal Of, of which the direction-cosin are $l\ m\ n$,

$$\frac{l^2}{f^2-a^2}+\frac{m^2}{f^2-b^2}+\frac{n^2}{f^2-c^2}=0.$$

[1] *Philos. Magazine*, ser. 3: 1837, vol. 11, pp. 461, 537; 1838, vol. 12, pp. 73, 3

rence, if $x\,y\,z$ be the co-ordinates of f, and the length of Of be de-
ll by r, we have

$$r=f:\ x=lr,\ y=mr,\ z=nr.$$

ubstituting in the above equation, we get

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2}=0.$$

his is the equation of the locus of the points f, or of the pedal o
ay-surface: the velocity of normal propagation of a ray-front along
radius vector of the surface is measured by the length of the radius
or.

. *The polar reciprocal of the ray-surface belongs to the same family*
ce of wave-slowness or index-surface.

e radius of a concentric reciprocating sphere being taken as unity, the
$\xi\ \eta\ \zeta$ which corresponds to the ray-front will lie in the front-normal Of
distance $\frac{1}{Of}$ from the origin.

nce if $x\,y\,z\,r$ refer to the point f, and $\xi\ \eta\ \zeta\ \rho$ to the pole of the ray-
, we have

$$\frac{x}{r}=\frac{\xi}{\rho};\ \frac{y}{r}=\frac{\eta}{\rho};\ \frac{z}{r}=\frac{\zeta}{\rho};\ r=\frac{1}{\rho}.$$

bstituting these values in the equation of the locus of the points f,
nd for the equation of the polar reciprocal of the ray-surface

$$\frac{\xi^2}{\frac{1}{\rho^2}-a^2}+\frac{\eta^2}{\frac{1}{\rho^2}-b^2}+\frac{\zeta^2}{\frac{1}{\rho^2}-c^2}=0$$

or

$$\frac{\frac{1}{a^2}\xi^2}{\rho^2-\frac{1}{a^2}}+\frac{\frac{1}{b^2}\eta^2}{\rho^2-\frac{1}{b^2}}+\frac{\frac{1}{c^2}\zeta^2}{\rho^2-\frac{1}{c^2}}=0$$

his is a surface of the same family as the ray-surface, being derived
the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ in the same way that the ray-surface
is derived from the indicatrix $a^2x^2+b^2y^2+c^2z^2=1$.
he surface has been distinguished by Hamilton as the *surface of*
-slowness, and by Mac Cullagh as the *index-surface*.[1]

[1] *Trans. Roy. Irish Acad.*: 1837, vol. 17, p. 142; 1839, vol. 18, p. 38.

41. The optic bi-normals (*primary optic axes*).

From Article 25 it follows that if the section of the indicatrix b plane having the direction of the ray-front is a circle, *every* point R this section corresponds to a ray having the same direction of front the same resolved velocity normal to the front, and therefore the sa position of front. The normals of the indicatrix at the points R, and th the planes of polarisation of the corresponding rays, may have azimuth whatever.

That there are at least two directions of central section of the indica for which the curve of section is a circle is seen as follows:—the sect of the indicatrix by any plane OBP (Fig. 15), passing through the m axis OB, is symmetrical both to the line $BO\overline{B}$ and the plane AOC, therefore to $PO\overline{P}$, P being a point of intersection of the curve with plane AOC: hence the section is in general an ellipse of which OB.

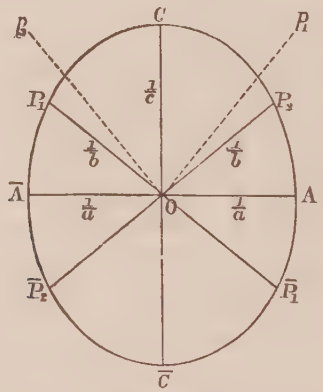

Fig. 15.

are the axes. If the direction OP be so taken in the plane AOC the radius vector OP is equal to OB, which is always possible since is intermediate in length between OA and OC, the axes of the ell become equal and the ellipse becomes a circle.

There are only two directions for which a radius vector OP of the ell AOC has the value OB or $\frac{1}{b}$. If $x\ 0\ z$ are the co-ordinates of P,

$$a^2x^2+c^2z^2=1, \text{ and } x^2+z^2=\frac{1}{b^2}: \quad \text{hence} \qquad \frac{x^2}{b^2-c^2}=\frac{z^2}{a^2-b^2}.$$

If $l\,0\,n$ be the direction-cosines of Op, the perpendicular to the line OP,

$$\frac{l}{n} = -\frac{z}{x}, \text{ and } \frac{l^2}{a^2-b^2} = \frac{n^2}{b^2-c^2}.$$

The angle pOC is given by the relation $\tan pOC = \pm\sqrt{\frac{a^2-b^2}{b^2-c^2}}$.

The two directions Op_1 Op_2 thus possess certain optical characters which calcite and analogous crystals only belong to that single direction which termed the optic axis: for ray-fronts normal to either Op_1 or Op_2 may be any azimuth of plane of polarisation whatever, and their velocities olved normally to the direction of front are equal. By reason of this logy, the directions Op_1 Op_2 have likewise been termed optic axes.

But the front-normals Op_1 Op_2 not being coincident with the corresponding rays, for they are not axes of the indicatrix, the directions Op_1 Op_2 not possess all the characters which belong to the optic axis of a axal crystal; to distinguish them from the directions Os_1 Os_2, which e been termed secondary optic axes, they have received the name *mary Optic Axes*: they have also been termed by Sir William milton *Lines of Single Normal-Velocity*.

Where precision of thought and language is necessary, the lines may veniently be termed the *Optic Bi-normals*, for they are directions which a line is doubly the central normal of a ray-front: the term correlative to bi-radial, and such a bi-normal cannot be confused with t of a three-dimensional curve.

112. *There cannot be more than one pair of optic bi-normals.*

It has already been proved that the only bi-normals in the plane AOC Op_1 Op_2: it remains to prove that there are no other bi-normals in direction whatever.

From Article **33** we have for the relation between $l\,m\,n\,f$ the equation

$$\frac{l^2}{f^2-a^2} + \frac{m^2}{f^2-b^2} + \frac{n^2}{f^2-c^2} = 0,$$

or $l^2(f^2-b^2)(f^2-c^2)+m^2(f^2-c^2)(f^2-a^2)+n^2(f^2-a^2)(f^2-b^2)=0.$

Since $a\,b\,c$ are in descending order of magnitude, the expression on the -hand side of the last equation is positive, and therefore cannot be , if f has any value greater than a or less than c; as is otherwise dent from the fact that $f = \frac{1}{OR}$, where R is a point on the indicatrix: ace no value of f greater than a or less than c can make the expression o.

Further, if m is distinct from zero, the above expression is necessari negative when $f = b$: hence it changes sign and passes through a ze value as f decreases from a to b, and again as f decreases from b to If m is distinct from zero, the two values of f^2 which satisfy the abo equation are thus unequal.

Hence the bi-normals can only lie in the plane AOC.

Since OB is normal to the plane of polarisation for any ray-front of whi the normal lies in the plane AOC, one root of the equation correspondi to such a ray-front is always $f=b$, and this must be the value of the equ roots: the directions of the bi-normals may therefore be found direct from the general equation (Art. 33) as follows:—

For any front-normal in the plane AOC, $m=0$, and the values of f^2 a given by the equation $\frac{l^2}{f^2-a^2} + \frac{n^2}{f^2-c^2} = 0$: hence, the directions of t front-normals in the plane AOC for which $f = b$ are given by the equati $\frac{l^2}{a^2-b^2} = \frac{n^2}{b^2-c^2}$; which is identical with the equation of last Article.

43. *The direction of a line Of being defined by its inclinations π_1 π_2 the bi-normals Op_1 Op_2, to find the transverse planes of the two ra of which the corresponding fronts are perpendicular to the given line.*

Let $[p_1]$ $[p_2]$ be the circular sections of the indicatrix perpendicul to the bi-normals Op_1 Op_2 respectively, and let $[f]$ be the central secti of the indicatrix parallel to the given direction of ray-front (Fig. 1

Let $[f]$ intersect $[p_1]$ $[p_2]$ in E_1 E_2 respectively.

All radii vectores in the two circular sections being equal, $OE_1 = OE$ and the axes OR OT of the elliptical section $[f]$ are therefore the i ternal and external bisectors of the angle E_1OE_2.

By Art. 26 the two rays Or Ot corresponding to the front-normal Of s in the planes fOR fOT respectively.

Again, Of is perpendicular to both OE_1 and OE_2;

Op_1 Op_2 are perpendicular to OE_1 and OE_2 respectively.

Hence OE_1 is perpendicular to both Of and Op_1, and therefore to th plane fOp_1;

OE_2 is perpendicular to both Of and Op_2, and therefore to their pla fOp_2.

The planes fOE_1 fOp_1 are thus at right angles: likewise the planes fO fOp_2.

Hence the planes fOR fOT which bisect the angles between fOE_1 a fOE_2, also bisect the angles between fOp_1 fOp_2, the planes which pa

through Of and the bi-normals Op_1 Op_2. In other words each of the perpendicular planes, which bisect the angles between the two planes passing through Of and one or other of the bi-normals, contains one of the rays of which the front is normal to Of, and is the *transverse* plane of the contained ray. As already pointed out in Art. **26**, it is the transverse planes, not the planes of polarisation, of the two rays corresponding to a single direction of front-normal which are perpendicular to each other.

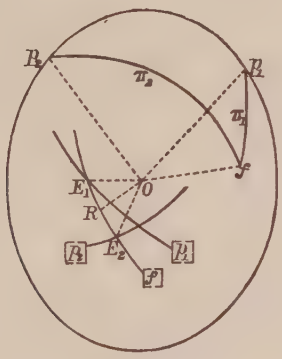

FIG. 16.

44. *The direction of a line Of being defined by its inclinations π_1 π_2 to the bi-normals Op_1 Op_2, to find f_1 f_2 the velocities of normal-transmission of the two ray-fronts which are perpendicular to the given line.*

From the equation of Art. **33**, it follows that

$$f_1^2+f_2^2=l^2(b^2+c^2)+m^2(c^2+a^2)+n^2(a^2+b^2)$$
$$f_1^2f_2^2=l^2b^2c^2+m^2c^2a^2+n^2a^2b^2.$$

Hence, proceeding by the method of Art. **20**, it may be shown, having due regard to the relative magnitudes of f_1 and f_2, that

$$f_1^2-f_2^2=(a^2-c^2)\sin\pi_1\sin\pi_2, \text{ and that}$$
$$2f_1^2=a^2+c^2+(a^2-c^2)\cos(\pi_1-\pi_2)$$
$$2f_2^2=a^2+c^2+(a^2-c^2)\cos(\pi_1+\pi_2):$$

whence

$$f_1^2=a^2\cos^2\frac{\pi_1-\pi_2}{2}+c^2\sin^2\frac{\pi_1-\pi_2}{2},$$

$$f_2^2=a^2\cos^2\frac{\pi_1+\pi_2}{2}+c^2\sin^2\frac{\pi_1+\pi_2}{2}.$$

45. The bi-radial cone (*the cone of front-normals which correspond to the rays transmissible along a bi-radial*).

We have seen (Article **11**) that in general for any given direction of ray Or there are four corresponding points of the indicatrix on two diameters OR_1 OR_2 of the conjugate plane. Also (Article **27**), there are two corresponding front-normals Of_1 Of_2, lying in the planes OrR_1 OrR_2 and perpendicular to OR_1 OR_2 respectively. But we have seen that in the case of a bi-radial Os_1, every point S on the conjugate section corresponds to a ray transmissible along that direction with the velocity b. Further, Os_1 is not an axis of the indicatrix, and thus is only coincident with the corresponding central normal On for the two rays transmissible in the direction Os_1 which correspond to the two points B $\bar{B}$ in which the axis $BO\bar{B}$ meets the conjugate section.

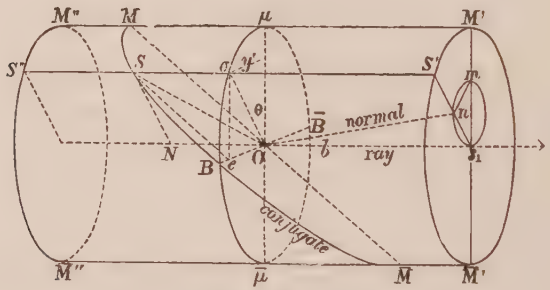

Fig. 17.

If $M''\bar{M}'$ (Fig. 17) be the right circular cylinder touching the indicatrix in the curve $MS\bar{M}$, in which the plane conjugate to the bi-radial Os_1 meets the indicatrix, and $M'S'\bar{M}'$ be a basal section perpendicular to the axis, the front-normal On, corresponding to any point S on the curve, is in the plane $SS's_1$ passing through the axis of the cylinder, and is at right angles to OS. Hence, as the point S moves round the curve $MB\bar{M}$, the corresponding normal On describes a cone of which the bi-radial Os is an edge, for it corresponds to the points B $\bar{B}$ on the curve: the cone may be conveniently designated a *bi-radial cone*. In the next Article it will be shown that the bi-radial cone intersects the base of its corresponding cylinder in a circle.

Corollary. Since the front corresponding to a ray touches the ray

ace at the point where the ray meets it, a cone of tangent planes can rawn at each of the points in which a bi-radial meets the ray-surface.

3. *A plane perpendicular to a bi-radial intersects the cone of corresponding front-normals in a circle.*

irst proof.

aking the base of the cylinder at such a distance from O that $Os_1 = b$, have $ns_1 = Os_1 \tan nOs_1$ (Fig. 17), for the angle ns_1O is by construc- a right angle.

so, by construction, $SN = \dfrac{1}{Os_1} = \dfrac{1}{b}$.

ence $\tan nOs_1 = \tan OSN = \dfrac{ON}{NS} = b \cdot ON$,

$$ns_1 = b^2 \cdot ON.$$

t $B\mu\bar{B}$ be a section of the cylinder parallel to the base, and let the line ntersect the curve μB in the point σ: further let MOs_1 be the sym- al plane of the indicatrix which contains the axes OA OC, and Om be irection of the front-normal corresponding to the point M on the atrix.

N and σO are equal and parallel, since they are both normal to the of the cylinder and in a plane containing it: hence $ON = S\sigma$.

raw Se σe perpendicular to the axis OB; and let the angle between onjugate section and the base of the cylinder be ϕ: then

$$Se\sigma = MO\mu = mOs_1 = \phi.$$

et the angle ns_1m be denoted by θ: to determine the relation between nd the angle θ, we thus require to express ON or $S\sigma$ in terms of the e ns_1m or $\sigma O\mu$.

e have $S\sigma = \sigma e \tan\phi = O\sigma \cos\theta \tan\phi = \dfrac{1}{b}\cos\theta \tan\phi$.

ence $ns_1 = b^2 \cdot ON = b^2 \cdot S\sigma = b \cos\theta \tan\phi = s_1m \cos\theta$.

he angle mns_1 is thus a right angle; and the locus of n is a circle ing through the point s_1, and having ms_1 for diameter.

econd proof.

et $x'y'z'$ be the co-ordinates of any point S on the section conjugate e bi-radial Os_1: by Article 18

$$\frac{x'^2}{c^2(b^2-c^2)} = \frac{z'^2}{a^2(a^2-b^2)}.$$

ow y', being the perpendicular from S or σ to the plane $M\mu M's_1$, is equal

$$\sigma O \sin \mu O\sigma = \frac{\sin\theta}{b}.$$

Finding x' z' also in terms of θ by means of the above equation, we ha

$$x'^2 = \frac{b^2-c^2}{a^2-c^2}\frac{\cos^2\theta}{a^2}; \qquad z'^2 = \frac{a^2-b^2}{a^2-c^2}\frac{\cos^2\theta}{c^2}.$$

Also $s_1n^2 = On^2 - Os_1^2$.

Draw s_1f parallel to OS and therefore perpendicular to On:

then $On = \frac{Os_1^2}{Of} = b^2 \cdot OS$, for $Of = \frac{1}{OS}$ (Art. **23**).

Hence $s_1n^2 = b^4OS^2 - b^2$.

$= b^4(x'^2+y'^2+z'^2) - b^2$

Substituting the above values of x' y' z' in terms of θ, we get

$$s_1n = \frac{b\cos\theta}{ac}\sqrt{(a^2-b^2)(b^2-c^2)}, \text{ or}$$

$s_1n = k\cos\theta$, where k is constant for all values of θ.

For $\theta = 0$, n takes the position m; hence $s_1n = s_1m\cos\theta$, as before.

Also $s_1m = k = \frac{b}{ac}\sqrt{(a^2-b^2)(b^2-c^2)}$.

47. *Aperture of the bi-radial cone.*

If the angle mOs_1 be termed the *aperture* of the cone, the aperture given by the relation

$$\tan mOs_1 = \frac{s_1m}{s_1O} = \frac{1}{ac}\sqrt{(a^2-b^2)(b^2-c^2)}.$$

The angle mOs_1 is the angle between Os_1, a bi-radial, and Om, a l perpendicular to the plane which is conjugate to that bi-radial; a re tion by means of which the above value may likewise be obtained.

48. *Polarisation of the ray corresponding to a given front-normal the bi-radial cone.*

For the ray transmissible along Os_1 which has On for its front-norm the normal of the plane of polarisation is SN or ns_1: hence the plane polarisation of that ray Os_1 which has On for its front-normal, meets base of the cone in a line parallel to the line nm, or in other words in line which joins s_1 to the other extremity of that diameter of the ci which passes through n.

49. **The bi-normal cone** (*the cone of rays corresponding front which is perpendicular to a bi-normal*).

In general (Article **26**), if OR OT (Figs. 13, 18) be the axes of a cen section of the indicatrix, the points R T correspond to rays Or Ot hav fronts in the same direction, namely parallel to the plane ORT: a if Of be the normal to the fronts, the rays Or Ot lie in the planes f fOT and are perpendicular to the lines which are normal to the indica

R and T respectively. But we have seen that all points on the cular section perpendicular to a bi-normal Op_1 correspond to rays ring the same position and direction of front, the latter being parallel the circular section: further Op_1 is not an axis of the indicatrix, and s is only coincident with the corresponding ray for the two points B on the circular section.

Hence, as the point R moves round the circular section of the indica-, the ray Or, which is always in the plane ROp_1, describes a cone of ich the bi-normal Op_1 is an edge, for it corresponds to the points $B\ \overline{B}$ the curve. The cone may be conveniently designated a *bi-normal* .

Corollary. Since every front touches the ray-surface where the cor-ponding ray meets it, a bi-normal is perpendicular to a plane which aches the ray-surface in a closed curve. In the next Article it will be wn that this curve is a circle.

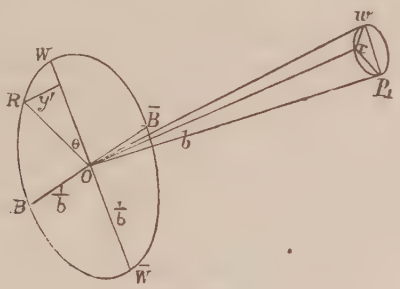

FIG. 18.

50. *A plane perpendicular to a bi-normal intersects the cone of corresponding rays in a circle.*

Let $W\overline{W}$ (Fig. 18) be the points where the circular section inter-cts the plane AOC: let $Op_1 = b$, and R be any point on the circular tion: the plane ROp_1, containing the ray Or corresponding to the int R, will intersect a plane, drawn through p_1 parallel to the circular tion, in a line p_1r parallel to OR; similarly, if Ow be the ray corres-nding to the point W, p_1w is parallel to OW: hence the angle rp_1w = angle W. Denote it by θ.

Also

$$p_1r^2 = Or^2 - Op_1^2$$
$$= a^4x'^2 + b^4y'^2 + c^4z'^2 - b^2,$$ if $x'y'z'$ be the co-ordinates of the nt R (Article 4).

But y' being the perpendicular from R on the plane WOp_1 or AOC

we have $$y' = OR \sin WOR = \frac{\sin\theta}{b}.$$

Also, since R is on a circular section,

$$\frac{x'^2}{b^2-c^2} = \frac{z'^2}{a^2-b^2} \text{ (Article 41)}$$

Finding x' and z' in terms of θ by means of this equation and th relation $a^2x'^2+b^2y'^2+c^2z'^2=1$, we get

$$x'^2 = \frac{b^2-c^2}{a^2-c^2}\frac{\cos^2\theta}{b^2} \qquad z'^2 = \frac{a^2-b^2}{a^2-c^2}\frac{\cos^2\theta}{b^2}$$

Substituting these values of x'^2, y'^2, z'^2 in the equation for p_1r',

we have $$p_1r = \frac{\cos\theta}{b}\sqrt{(a^2-b^2)(b^2-c^2)}, \text{ or}$$

$p_1r = l\cos\theta$, where l is a constant for all values of θ.

For $\theta = 0$, r takes the position w:

hence $p_1r = p_1w\cos\theta$; and $p_1w = \frac{1}{b}\sqrt{(a^2-b^2)(b^2-c^2)}$.

The angle p_1rw is thus a right angle; hence the locus of r is a circ passing through the point p_1 and having p_1w for diameter.

The section of the cone of rays by a plane perpendicular to that b normal which is the front-normal for the rays is therefore a circle.

51. *Aperture of the bi-normal cone.*

If the angle wOp_1 be termed the aperture of the cone, the aperture given by the relation

$$\tan wOp_1 = \frac{p_1w}{p_1O} = \frac{1}{b^2}\sqrt{(a^2-b^2)(b^2-c^2)}.$$

The angle wOp_1 is equal to the angle between the radius-vector O and the normal of the indicatrix at W, for Op_1 and Ow are respective perpendicular to OW and the normal at W; a relation by means of whi the above value may likewise be obtained.

52. *Polarisation of the rays of the bi-normal cone.*

The plane of polarisation of the ray Or is a plane perpendicular to t transverse plane Op_1r. The line Op_1, being perpendicular to the pla p_1rw, is perpendicular to the line joining p_1 to r, the other extremity that diameter of the circle p_1rw which passes through r; the line rp_1, bei perpendicular to both p_1O and p_1r, is perpendicular to the plane Op containing them: as any plane passing through p_1r, or its parallel rw, likewise perpendicular to the plane Op_1r, the plane Orw is the plane polarisation of the ray Or.

3. Representative surfaces derived from the Indi-rix.

The characters of a ray of light transmitted in a crystal may also pressed by reference to corresponding points on the polar reciprocal e indicatrix relative to a concentric sphere: this surface is an oid represented by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (p. 381 and Fig. 19).

OR, a radius vector of the indicatrix, be normal to a tangent of the polar reciprocal of the indicatrix, meeting the plane in a point R·OM=1, if the radius of the reciprocating sphere be unity. If P point in which the tangent plane perpendicular to OR touches the reciprocal, and PG be the normal of the latter surface at the point lines PG PO lie in the plane $RNOr$: let PG intersect the ray the point G. If m be the point in which OP intersects the plane touches the indicatrix at R, OP·$Om=1$, and OP is thus the e of RN. Hence, to every point P on the polar reciprocal of the trix corresponds a ray Or: it lies in the plane $PGOr$, and is per-cular to OP: its velocity of transmission is measured by OP: its erse plane is $PGOr$: the ray-front intersects the transverse plane perpendicularly in a line parallel to PG, and its velocity of normal-mission is measured by PG.

Von Lang[1] has pointed out that if a surface be derived from the id $a^2x^2+b^2y^2+c^2z^2=1$ by elongating each radius vector until the new is measured by the nth power of its original value, the derivative e may likewise be used for the geometrical representation of the ters of transmitted rays. This result can be generalised still farther, ows:—

$\phi^{-1}(r)$ be any function of r, which always increases and decreases or *vice versâ*: it will have an apsidal (*i.e.* maximum or minimum) at the same time as r. If then a new surface be derived by ting each radius vector r of the indicatrix to a length ρ, determined relation $\rho=\phi^{-1}(r)$ or $r=\phi(\rho)$, a central section of the new surface ve its apsidal diameters in exactly the same directions as those of tion of the indicatrix by the same plane. If ρ_1 ρ_2 be the half-s of the new diameters, the corresponding ray-fronts are respectively ances $\frac{1}{r_1}$ and $\frac{1}{r_2}$ or $\frac{1}{\phi(\rho_1)}$ and $\frac{1}{\phi(\rho_2)}$ from the central section; the ray-e itself is the envelope of these planes.

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[1] *Sitz. Ak. Wien*, 1861, vol. 43, sec. 2, p. 645.

The general equation of the new surface is easily found :—

If r be the length of a radius vector of the indicatrix and $l\ m\ n$ be direction-cosines, $a^2l^2 + b^2m^2 + c^2n^2 = \frac{1}{r^2}$; $\xi\ \eta\ \zeta$ being the co-ordinate the corresponding point on the new surface, $\xi = l\rho$, $\eta = m\rho$, $\zeta = n\rho$:

$$\text{whence } a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 = \frac{\rho^2}{r^2}$$

$$\text{or } a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 = \frac{\rho^2}{[\phi(\rho)]^2};$$

which is the required equation.

Fresnel's "surface of elasticity" is the particular case in w $\phi(\rho) = \frac{1}{\rho}$, for the equation then becomes $a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 = (\xi^2 + \eta^2 + \zeta$

For the "surface of elasticity," the transverse planes of the rays co ponding to a given direction of ray-front pass through the apsidal meters of a central section, as in the case of the indicatrix, but the dist of the ray-front corresponding to a semi-diameter of length ρ_1 is $\frac{1}{\rho_1}$ as in the indicatrix, but $\frac{1}{\phi(\rho_1)}$ or ρ_1.

The corresponding ray is only perpendicular to the correspon normal of the representative surface in the case of the indicatrix : in e case, however, the normal of the representative surface lies in the p passing through the corresponding diameter and the front-normal : fo curves of intersection of the two surfaces by the given plane have pa tangents at the extremities of their maximum and minimum diame Hence, as in the case of the indicatrix, the plane passing throu diameter and a normal of the surface at the extremity of the diamet the *transverse* plane of the corresponding ray.

(*c*) In exactly the same way a series of surfaces can be derived the polar reciprocal of the indicatrix.

The above generalisation serves as a reminder that there is not *sarily* a simple relation between a surface of geometrical represent and the characters of the ether.

...TION V.—VARIOUS OPTICAL RELATIONS WHICH ARE INDEPENDENT OF THE PHYSICAL CHARACTER OF THE PERIODIC CHANGE.

. In Section II we have shown that after the discovery of the ...risation of light by reflection by Malus in 1808, and of the corres... ...dence of optical and morphological symmetry by Brewster in 1819, ...true laws of transmission of light in biaxal crystals must soon have ...n suggested, independently of any hypothesis as to the physical character ...he periodic change: in fact, their enunciation by Fresnel in 1821 was ... two years later than Biot's discovery of two empirical laws by which ... accuracy of a geometrical representation could be tested. If the ...h of the construction given by Huygens for the case of calcite is ...owledged, the suggestion presents itself as soon as the planes of ...risation of the two rays transmissible in any direction in a crystal of ...ite are represented by their normals.

...n the present Section we proceed to indicate very briefly, for the ...venience of the student, various other important relations, which, ...gh really independent of any hypothesis as to the nature of the ...odic change, are usually imagined and expressed as belonging to an ...tic ether. It will at the same time be shown that the form of the ray-...face for biaxal crystals is not merely suggested by a geometrical ...eralisation as a tentative one, but is a necessary consequence of the ...rence of symmetrical development of the same physical characters, ...tever they may be, which originate the sphere and spheroid of a ...axal crystal: it will further be shown that the same form of the ray-...face would result from the general features of perpendicularly transverse ...ulations, and be independent of the real nature of the periodic change.

Preliminary algebraical expression for the transmission of a ray of ordinary light.

. It will be convenient, in the first place, to find a mathematical ...ression connecting the magnitude of the disturbance or change of state ...ny point in a ray of ordinary light of simple colour with the position of ... point, the time, and the period of the vibration. For this purpose ... necessary to make an assumption as to the law of the change: the ...plest which can be made is that at any point of a ray of ordinary light

of simple colour the variation of the state with the time follows the law as the variation of position of an isochronous pendulum.

It will be found that, for a ray of simple colour, the expression

$$y = a\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\}$$

is one which satisfies this condition and is consistent with all e ments *as yet referred to;*

x denoting the distance of any point in the ray from a fixed point

y the magnitude of the disturbance or change of state at the poin the time t;

v the velocity of transmission,

λ the wave-length,

a and α two constants for all values of x and t:—

1. At a given *point*, indicated by its distance x from the origi change of state varies periodically with the time t: the same value and therefore the same change of state, recurs whenever the expr $\frac{2\pi}{\lambda}(vt-x)+\alpha$ increases by 2π, that is when t increases by the co interval $\frac{\lambda}{v}$. The same change of state recurs, but with opposite whenever the expression $\frac{2\pi}{\lambda}(vt-x)+\alpha$ increases by π, that is w increases by half the above interval.

2. At a given *instant*, indicated by the time t, the change of st the same in magnitude and sign for all points separated from each by the distance λ: it is the same in magnitude and opposite in sign f points separated from each other by half that distance.

3. The relation between y and t is identical with the relation be the position of an isochronous pendulum and the time.

a, being the maximum value of y, is the amplitude of the vibratio $\frac{2\pi}{\lambda}(vt-x)+\alpha$ being the phase of the vibration at the point x a time t, α is the phase of the vibration at the origin ($x=0$) at the from which the time is measured ($t=0$).

The period of the vibration being independent of the amplitud law is consistent with the independence of colour and intensity.

Conversely, if the magnitude of the change of state at each point line is given by the expression $y=a\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\}$, and the ch is of the physical character which belongs to light, a ray of light of s

lour is passing along the line with a velocity v: the intensity corres-nds to the amplitude a, the colour to the period $\frac{\lambda}{v}$, while the phase of e vibration at the origin at the initial epoch is α.

From analogy with sound, we may tentatively assume that the intensity the light corresponding to this simple change of state is measured by e square of the amplitude.

sultant effect of the simultaneous transmission of two or more such rays along the same line.

3. The fact of the periodicity of the change was deduced from periments relative to the mutual interference of rays of light: it is easily n that the above expression for the change, combined with the principle superposition, is consistent with the observed phenomena from which it s deduced.

(a) *If the component rays have the same wave-length and velocity.*

. For let two rays of the same simple colour, transmissible along a en line with the same velocity v, be represented respectively by the pressions

$$y = a \sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\}$$

$$y = b \sin\left\{\frac{2\pi}{\lambda}(vt-x)+\beta\right\}:$$

both are transmitted simultaneously, the principle of superposition uires the resultant change to be determined by the expression

$$y = a \sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\} + b \sin\left\{\frac{2\pi}{\lambda}(vt-x)+\beta\right\}.$$

If the terms can be added together *in the same way as numerical antities of a single kind,*

$$y = (a\cos\alpha + b\cos\beta)\sin\frac{2\pi}{\lambda}(vt-x) + (a\sin\alpha + b\sin\beta)\cos\frac{2\pi}{\lambda}(vt-x)$$

$$= c \sin\left\{\frac{2\pi}{\lambda}(vt-x)+\gamma\right\},$$

if $$c^2 = a^2 + b^2 + 2ab\cos(\alpha-\beta)$$

and $$\tan\gamma = \frac{a\sin\alpha + b\sin\beta}{a\cos\alpha + b\cos\beta}.$$

Hence the resultant effect of the two rays is identical with that of a gle ray transmitted along the same line with the same velocity and the me wave-length (and thus of the same colour), but having an intensity nd an original phase γ. And the intensity of the resultant ray depends

not only on the intensities of the component rays but on their differen of phase at the same point at the same instant: if a and b are equal a α differs from β by any odd multiple of π, the intensity of the sing resultant ray is constantly zero.

2. It may in this way be shown that the resultant effect of the simu taneous transmission of any number of such rays of the same simp colour along the same line with the same velocity is identical with th of a single ray of the same colour and velocity, and having a determinat phase and intensity.

(*b*) *If the component rays differ in wave-length or velocity.*

On the other hand, if the component rays differ either in velocity wave-length, the resultant effect is not that of a single ray of simple colo the resultant effect is still expressed by

$$a \sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\}+b \sin\left\{\frac{2\pi}{\lambda'}(v't-x)+\beta\right\};$$

but the expression cannot take the simpler form $c \sin\left\{\frac{2\pi}{\lambda''}(v''t-x)+\gamma\right.$ in which c and γ are both constants: indeed, the resultant effect is r periodic at all unless the ratio $\frac{v}{\lambda}:\frac{v'}{\lambda'}$ is commensurable.

Kinematical representation of the periodic change at any point of such a r

4. Whatever be the physical character of the periodic change at a point of a ray of light, the state at any point P at a given inst may thus (consistently with any facts as yet indicated) be represented the above expression

$$y=a \sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\}:$$

this algebraical expression may in turn be represented geometrically; magnitude y being represented by the distance of a point p from the po P, and the distance being considered positive or negative according to t direction in which it is measured. The phenomena of interference, fr which the above expression has been deduced, merely require the directi in which the line Pp is measured to be necessarily the same for all poi of the same ray, and for all interfering rays transmitted along the same li

This mode of representation in no way assumes that the *actual* char of state at the point P is a to-and-fro motion of a particle of ether in t arbitrary line Pp; the direction of the line Pp is required to be const merely to secure that the changes, if they have any directional charac

all, may be added together like simple numerical quantities of the me kind: that the change is really directional in character may be in- red from the fact that it is being transmitted in a definite direction rough the medium. In exactly the same way, the transmission of a ray light along a line is sometimes conveniently represented (in the discus- n of aberration, for instance) by the transmission of a point along the e with constant velocity, although light is certainly not due to the nsmission of a particle along the direction of the ray.

Preliminary algebraical expression for the transmission of a ray of plane-polarised light.

5. It was found by Fresnel, in conjunction with Arago, that two rays of ne-polarised light, if their planes of polarisation are parallel, *may* tually interfere in exactly the same way as ordinary light: hence, as as this experiment goes, the periodic change at any point of a plane- arised ray can be represented in exactly the same way as for ordinary ht; the only difference being that while an ordinary ray is so far analo- us to a circular cylinder that its characters are identical on all its sides, plane-polarised ray is analogous to an elliptical cylinder to the extent t the properties of the ray are dissimilarly symmetrical relative to two rpendicular planes (pages 288 and 293).

If all the characters of a plane-polarised ray can be accounted for by ch a kinematical representation as is mentioned above, the line Pp must either in the plane of polarisation or the transverse plane; but it may ve any inclination whatsoever to the ray, so long as for two interfering ys the direction is identical.

re general representation of the periodic change at any point of an ordinary or plane-polarised ray.

5. Since, as far as the above experiments are concerned, the inclina- n of the direction Pp to the line of transmission of either a plane- arised or an ordinary ray, may be any whatsoever, it follows that the nge may really not be simple, but *multiple* in direction; assuming t each transmitted periodic change will interfere for itself, as if those ving other directions did not exist.

In fact, it will be seen that the periodic change may likewise be repre- ted by the composite expression

$$y = a_1 \sin\left\{\frac{2\pi}{\lambda}(vt - x) + \alpha_1\right\} + \ldots\ldots\ldots + a_n \sin\left\{\frac{2\pi}{\lambda}(vt - x) + \alpha_n\right\}$$

sisting of any number of terms: for each separate term, independently

of its directional relations, resumes its original value at distances along t ray separated from each other by the common length λ, or at times sep rated from each other by the common period $\frac{\lambda}{v}$: hence, if two rays nihilate each other under given circumstances, annihilation will again ta place if one of the rays is moved *parallel to itself* through the distance along its line of transmission.

And it is important to remark that as each term recurs *individua* after the same interval of time or distance, the whole expression likew recurs and has the same total value, even if the terms are *not* subject the same law of addition as simple numerical quantities.

It will also be obvious on reflection that any ray which is within reach of experiment is necessarily composite as regards the origin of vibration, even if it be simple as regards its colour: the luminous sou is not a geometrical point, but a surface of considerable dimensions as co pared with the wave-length of a ray of light; hence the periodic chan of which the effects are observed at a given point of a line of transmissi is really of composite origin and due to the superposition of the perio changes transmitted from the points of a luminous area of appreci magnitude.

As for the difference between an ordinary and a plane-polarised ray, first suggestion which presents itself is that the latter is due to the tortion of the ordinary ray from which it was derived; just as an el tical cylinder may be derived from a circular cylinder by compression direction inclined to the axis.

Experimental discovery made by Fresnel and Arago.

7. (*a*) But Fresnel and Arago found that, when one of two interfe plane-polarised rays is turned through a right angle round its directio transmission, the interference-effects completely disappear, whatever difference of phase of the two rays. Hence, with this relative posi of the planes of polarisation, the periodic change produced at any p by the transmission of one ray is in no direction coincident with a peri change produced by the transmission of the other ray; for as we seen (Art. 3), such coincidence would involve a variation of intensit the resultant effect: if this be granted, it follows that for a plane-polar ray the actual periodic change must be in only a single direction, and single direction must be perpendicular to the line of transmission; otherwise the two positions of the plane-polarised ray would give positions of the periodic change which would have a resolved par

mmon. Since the direction is single, it must be in one of the sym-tral planes of the ray: hence the direction of the actual periodic change perpendicular to the direction of transmission, and may be either in or rpendicular to the plane of polarisation: in either case it may be *repre-ted* by a line perpendicular to the plane of polarisation.

In the above experiments of Fresnel and Arago, the rays were allowed interfere during aerial transmission; it may reasonably be assumed, wever, that the same kind of symmetry with respect to two perpendi-ar planes obtains for a plane-polarised ray as transmitted within any rstalline medium: the assumption is not only reasonable on general ounds, but is consistent at once with all known experimental results and h the requirements of the most recent version of the elastic theory e also pages 293, 308). It is not the only assumption which can be made: esnel himself was led by the hypothesis of an incompressible elastic er to infer that a plane-polarised ray transmitted within a bi-refractive dium is in general symmetrical to only a single plane, perpendicular to plane of polarisation; he inferred, in fact, that the vibrations of the er lie in the transverse plane and are in general oblique, not perpendi-ar, to the direction of the ray. That Fresnel felt the unsatisfactory racter of the inference, in the absence of any experimental proof of obliquity, will be seen on reference to the original memoir.[1]

(*b*) If the two polarised rays which have been obtained from a ray of dinary light by means of a crystal of calcite are transmitted along the ne line, it is found that the resultant effect is again that of a single of ordinary light: hence we may infer that in ordinary light, as in ne-polarised light, the vibrations are perpendicular to the direction of nsmission of the ray.

Representation of the resultant effect of the simultaneous transmission ng the same line of two or more plane-polarised rays having different directions of planes of polarisation.

3. (*a*.) *If the component rays have the same wave-length and velocity.*) The periodic change at any point of a plane-polarised ray being kine-tically represented by a vibration perpendicular to the plane of arisation, let two rays be transmitted with the same velocity along the ne line, having different directions of the plane of polarisation: and, in first place, let the algebraical expressions for the corresponding nges be respectively

$$y = a \sin\left\{\frac{2\pi}{\lambda}(vt - x) + \alpha\right\} \text{ and } z = b \sin\left\{\frac{2\pi}{\lambda}(vt - x) + \beta\right\}.$$

[1] *Loc. cit.*; 1827, p. 158.

Assuming as before the principle of superposition, the effect of trar mitting both rays simultaneously will be represented by the motion of point of which the co-ordinates y and z, measured along the normals the planes of polarisation, are given by the expressions

$$y = a \sin\left\{\frac{2\pi}{\lambda}(vt - x) + \alpha\right\}$$

$$z = b \sin\left\{\frac{2\pi}{\lambda}(vt - x) + \beta\right\}.$$

Eliminating $\frac{2\pi}{\lambda}(vt - x)$, we find

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} - \frac{2yz}{ab}\cos(\alpha - \beta) = \sin^2(\alpha - \beta).$$

Hence the point, of which the position at any instant represents t resultant disturbance at that instant at a corresponding point on the li of transmission, describes in general an ellipse, of which the magnitu and position relative to the planes of polarisation of the original rays a independent both of x and t: all the ellipses are thus equal and parall and form a cylinder of which the base is elliptical, and the axis is in t direction of transmission. It will be found that the direction in which t point moves round the ellipse is determined by the relative phases of t two rays. The composite or resultant ray of light due to the co-exister of the original rays is said to be *elliptically polarised*; a ray of which t characters are related to a cylinder with elliptical base must differ fron ray of ordinary light, of which the characters are the same on all its side

(2.) If the rays have the same intensity, and their difference of phase measured by the angle between their planes of polarisation, $a = b$, and $\alpha -$ is equal to the angle between the directions of y and z: in this case t ellipse becomes a circle, and the cylinder becomes one with a circular ba The composite ray is then said to be *circularly polarised*. Such a ray similarly related to every plane passing through it, and yet differs fr one of ordinary light: for the motion of the representative point is r symmetrical to a plane, and the characters of the ray may conceival differ with the direction in which the circle is described by the ideal poi In fact, experimental methods enable us to distinguish, not only betwe a ray of ordinary light and one which is circularly polarised, but betwe two circularly polarised rays of which the motion of the ideal point is opposite directions.

(3.) If $\sin(\alpha - \beta) = 0$, that is to say, if the difference of phase is z or a multiple of π, the ellipse becomes one or other of the two straight li

$\left(\frac{y}{a}\pm\frac{z}{b}\right)^2=0$: hence the resultant ray is itself *plane-polarised*, the direc-
ion of the plane of polarisation depending on the ratio $a:b$, and thus
eing determined by the relative intensities of the two component rays.
onversely, such a single plane-polarised ray of simple colour is equivalent
i its effects to two such plane-polarised rays of the same simple colour,
ansmitted along the same line with the same velocity, and with their
anes of polarisation in *any* assigned directions. If the two assigned
rections be perpendicular to each other, and θ be the inclination of one
them to the plane of polarisation of the original ray supposed to be
presented by the expression

$$y=a\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\},$$

e two equivalent rays are represented respectively by the expressions

$$y=a\sin\theta\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\}$$

$$z=a\cos\theta\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\};$$

r the resultant effect of these two rays is such that

$$\frac{y}{z}=\tan\theta, \text{ a constant quantity,}$$

atever be the time or the position of the point in the line of trans-
ission.

(4.) Further, it will be seen that any number of such rays of the
me simple colour transmitted along the same line with the same velocity
t with different phases, amplitudes and planes of polarisation, will have a
sultant effect identical in general with that of a single elliptically polarised
y of the same simple colour, transmitted along the same line with the
me velocity. For let the simple rays be severally represented by the
pressions

$$y_1=a_1\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha_1\right\}$$

$$y_2=a_2\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha_2\right\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_n=a_n\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha_n\right\},$$

d let the inclinations of the respective planes of polarisation to a fixed

plane of reference through the line of transmission be $\theta_1\ \theta_2 \ldots\ \ldots\theta_n$. Eac single ray being equivalent in effect to two rays with perpendicular plan of polarisation, one of them coincident with the fixed plane of referenc the whole system of rays is equivalent to the following two systems:—

$$y = a_1 \sin\theta_1 \sin\left\{\frac{2\pi}{\lambda}(vt-x)+a_1\right\}+\ldots\ldots+a_n \sin\theta_n \sin\left\{\frac{2\pi}{\lambda}(vt-x)+a_n\right\}$$

$$z = a_1 \cos\theta_1 \sin\left\{\frac{2\pi}{\lambda}(vt-x)+a_1\right\}+\ldots\ldots+a_n \cos\theta_n \sin\left\{\frac{2\pi}{\lambda}(vt-x)+a_n\right\};$$

all the members of each of these systems having a common direction plane of polarisation.

As each system is equivalent to a single plane-polarised ray (Arts. and 5), the two systems are together equivalent in general to a sing elliptically polarised ray.

(5.) Whether the resultant ray be elliptically, circularly, or plan polarised, the resultant change has the same period as the change f each component ray, and is thus of unaltered colour.

(6.) At a given instant, the ideal points representing the state at points of the resultant ray lie on a spiral curve surrounding the elliptical circular cylinder, if the ray be elliptically or circularly polarised, and on undulating curve (the curve of sines) in the transverse plane, if the ray plane-polarised.

(*b.*) *If the component rays differ in wave-length or velocity.*

If the two component rays differ in wave-length or velocity of tra mission, the resultant effect is still represented by the combin expressions

$$y = a \sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\}$$

$$z = b \sin\left\{\frac{2\pi}{\lambda'}(v't-x)+\beta\right\};$$

but it is not periodic at all unless the ratio $\frac{v}{\lambda}:\frac{v'}{\lambda'}$ is commensurable: a even in that case the curve described by an ideal point is not a co section.

The resultant effect can only be that of a plane-polarised ray if the ra of y to z, and therefore of $\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\alpha\right\}$ to $\sin\left\{\frac{2\pi}{\lambda}(vt-x)+\beta\right.$ is independent of the time: but if either v or λ is different from v' or respectively, this constancy is impossible, whether the planes of polarisat of the original rays are real or imaginary.

Discrepancy of observed and calculated results.

9. But the above calculation of the resultant effect of the simultaneous nsmission along the same line of two plane-polarised rays of the same our with planes of polarisation at right angles to each other is in direct agreement with the experimental result recorded in Art. 7*b*, for the ult of superposition of the two plane-polarised rays obtained from an linary ray by means of a bi-refractive crystal is not an elliptically polarised , but a ray of ordinary light having identical characters on every side. are thus led to inquire how far the constancy of character of the periodic nges at points in the same ray has really been established by experiment.

In fact, the annihilation-effect (p. 286) of two rays of identical charac- has only been established for a transference of one of the rays through a tance of at most 50,000 wave-lengths: the wave-length in air for sodium- t being nearly $\frac{3}{5000}$ millimetres, the above distance is nearly 30 millimetres or about one inch: as light is transmitted through air at the rate 186,000 miles a second, a distance of one inch corresponds to the se of only $\frac{1}{11,784,960,000}$ th part of a second.

The discrepancy disappears if a ray is assumed to consist of a series of independent sets of waves of the same length.

10. For the sake of a numerical example, let us imagine that two given s are absolutely identical in character; that each ray consists of a series sections; that each section consists of at least a million similar waves, t that the waves of one section are absolutely independent of those of ery other, except that they have the same period and are transmitted with same velocity.

Let the constant sections of one ray be $A_1B_1\ B_1C_1\ C_1D_1 \ldots\ldots Y_1Z_1$, d the identical sections of the other ray be $A_2B_2\ B_2C_2\ C_2D_2 \ldots\ldots Y_2Z_2$: nsider the resultant effect of transmitting both heterogeneous rays simulneously along the same line.

(1.) If the initial points A_1 A_2 coincide, the vibrations are in unison at ery point of every section, notwithstanding the heterogeneity of each ray.

(2.) If the ray $A_2 \ldots\ldots Z_2$ be moved parallel to itself along its own ection through the distance $\frac{\lambda}{2}$, the two rays will annihilate each other all points where identical sections are superposed, but will in general to do so in the regions where different sections overlap; that is, for a

distance $\frac{\lambda}{2}$ at the end of every section. Hence at any given point the
will be annihilation while at least 999,999½ waves pass by, and more
less unison while half a wave is passing the same point.

(3.) In the same way, if the ray $A_2 \ldots\ldots Z_2$ be moved parallel to its
along its own direction through the distance 50,000½ wave-lengths, t
two rays will still annihilate each other at all points where identi
sections are superposed, but will in general fail to do so in the regio
where different sections overlap; that is, for a distance 50,000½ wav
lengths at the end of each section. Hence, at any given point, there w
be complete annihilation while at least 949,999½ waves pass by, and mo
or less unison while 50,000½ waves are passing the same point: in oth
words, instead of complete annihilation, there is more or less light duri
at most $\frac{1}{20}$th part of the time: the light will be apparently continuous, b
its intensity will not exceed the $\frac{1}{20}$th part of the maximum joint effect
the two rays. The variability of the periodic character will thus accou
for the appreciable diminution of the interference-effect when one of the ra
is moved parallel to itself through a considerable number of wave-lengti

In the following pages we shall only need to consider sets of waves b
longing to a single section of constant periodic character, and may th
proceed as if the constancy of character were really a property of t
whole ray.

*The same assumption accounts for the remarkable fact that rays of t
same simple colour, but obtained from different sources, cannot be made
annihilate each other.*

11. Hitherto, for simplicity, we have left unmentioned the remarkab
fact that rays of light of the same simple colour, whether ordinary or plan
polarised, cannot be made to annihilate each other if they have be
derived from different sources. This is quite inexplicable if a ray
assumed to have constancy of periodic character throughout its exten
but it is immediately accounted for by the assumption arrived at in th
preceding Article: if a ray consists of a series of *independent* sets
waves, it is physically impossible for two rays from different sources to
identical in their characters.

For a plane-polarised ray, only the amplitudes and phases will diff
in the different sets.

We have seen that two plane-polarised rays of constant periodic ch
racter throughout would give an elliptically polarised ray of which th
ellipses would have a definite magnitude and position dependent on the ar

udes and phases of the component rays: if each of the plane-polarised s, instead of being of constant periodic character throughout, consists independent sets of waves, the resultant effect will generally be a rapid cession of elliptically polarised sets, the magnitudes and positions of the ses changing as different sections of the plane-polarised rays become erposed; the resultant ray will thus generally be identical in character all its sides, as far as observation can detect.

only is the assumption of variability of periodic character necessary, t a constancy of periodic character could not be physically maintained.

2. A simple pendulum, disturbed and then set free to oscillate under constant action of gravity, soon comes to rest if allowed to communicate motion to a surrounding medium: to maintain the oscillations, the dulum requires to be repeatedly disturbed, and each impulse may change phase and amplitude, and possibly also the direction of the vibration. ne same way, the vibrations of character at the points of a luminous must be maintained by the repeated action of something analogous a impulsive force. It is impossible to imagine that the representative lse can always have the same magnitude and direction and occur at particular instant when the vibration is in a particular phase. Hence ibration must, of almost absolute necessity, be different in its ampli- phase, or direction, after every impulse.

rther, as already remarked in Art. 6, any luminous source available experiment is not a geometrical point, but an area of appreciable mag- de, and the resultant effect at any point is due to the superposition of effects of rays transmitted from every point of the luminous area: even were possible that the vibrations at a single point could be maintained tant in periodic character, it is inconceivable that the constancy of dic character could be maintained at points belonging to an appreci- area.

A representative force.

. In the case of a plane-polarised ray of constant character throughout art considered, the vibratory motion of the representative point p is thus ame for all points P in the line of transmission, and only the phase of ibration differs at different points at a given instant: hence the expres-

$y = a \sin \frac{2\pi}{\lambda} vt$, which represents the change of state at the time t at

oint for which $\frac{2\pi . x}{\lambda} - a = 0$, also represents the vibration at any other of the ray, if we have due regard in every case to the epoch from

which the time is measured. The general expression for the law of t change at any point of a plane-polarised ray has been deduced on the sumption that the variation of the state with the time is exactly t same as the variation of position of an isochronous pendulum; what is the same thing, of a particle of unit mass vibrating in a straig line and attracted towards an origin in the line by a force of whi the magnitude is proportional to the distance therefrom. For t velocity u of the attracted particle at the time t being $\frac{dy}{dt}$, the accelerat force at the same instant is $\frac{du}{dt}$ or $\frac{d^2y}{dt^2}$: by hypothesis the force is attracti and is measured by f^2 times the distance, or by $-f^2y$, where f is a consta quantity: hence $\frac{d^2y}{dt^2} = -f^2y$.

It is easily seen that $y = B\sin(ft + \beta)$, in which B and β are b independent of the time, is a solution of this differential equation: differentiating once we have $\frac{dy}{dt} = fB\cos(ft+\beta)$, and differentiating second time $\frac{d^2y}{dt^2} = -f^2B\sin(ft+\beta) = -f^2y$.

If the time be measured from an epoch of passage through the orig the constant β is zero and the expression becomes $y = B\sin ft$.

Hence in the case of plane-polarised light, the vibratory motion of representative point p, being expressed by the relation $y = a \sin \frac{2\pi}{\lambda} vt$, identical with that of a particle of unit mass attracted towards the origin a force which is measured by $\frac{4\pi^2v^2}{\lambda^2}$ times the distance.

Even if the actual change of state at the point P were an oscillat rotation of an ethereal particle about a diameter, as suggested by Rankin the above kinematical representation would still hold: in that case, direction of the line Pp would represent that of the axis of rotation of ethereal particle at P, and the distance Pp would represent the angu disturbance at the given instant.

Or again, the real change may be an electro-magnetic disturbance, wh ever that may be.

The representative force is dependent on the luminous source.

14. But it will be obvious on reflection that the relation between

[1] *Philos. Magazine;* 1853, ser. 4, vol. 6, p. 403.

nce of the ideal particle, and the ideal force which acting upon the ideal cle would cause a vibration isochronous with that of the periodic ge involved in the transmission of the given ray of light, is generally endent of the specific properties of the transmitting medium. The being $4\pi^2v^2 : \lambda^2$ depends only on the ratio $\lambda : v$, that is to say on the d of the vibration or the colour of the light. Now *simple* light ally retains its colour after transmission through any number of ent media; it is only in fluorescent bodies that the colour of the light e period of the change suffers alteration: whence we must infer the period of vibration at any point of a ray, and thus the ratio e ideal force to the distance, depends in general, not on the specific rties of the medium, but on the period of vibration of the change at minous source. The change of colour frequently observed after the ge of light through a medium is really due to the heterogeneity of lour of the original light, and to the change of relative *intensity* (not of vibration, or colour) of the component simple rays.

Further analogy with sound.

The same is true in the case of sound. Here again the trans- on of a simple note causes a periodic change which may be represented ically by the same expression $y = a \sin\left\{\frac{2\pi}{\lambda}(vt - x) + a\right\}$, and atically by the same to-and-fro motion of a particle attracted to an with a force measured by $\frac{4\pi^2v^2}{\lambda^2}$ times the distance: and the constant $4\pi^2v^2 : \lambda^2$ depends only on the period of the vibration or the note of und, and thus on the source of the sound, not on the properties of the itting medium. But the actual change of state at any point of a line smission of sound being known to be generally a to-and-fro motion article of the medium, the ideal particle and its motion may generally en to coincide with the real particle and its motion. Hence the ude of the representative force which acts on the ideal particle must confused with that of the elastic force which is evoked at the oint by the disturbance of the sound-transmitting medium: the ntative force depends on the period of vibration at the source; the force evoked by a given displacement depends on the specific ties of the medium: the resultant force acting on the real particle s, not only on the specific properties of the medium, but on the ed action of the luminous source.

The representative force in the case of the vibration of an elastic ether of wi the effective density depends on the direction of the vibration.

16. To take another example: in the latest hypothesis as to the proper of an elastic luminiferous ether, it is assumed that the actual and effec elasticity of both volume and figure and the actual density of the ether the same for all directions in a biaxal crystal, but that the effec density varies with the direction of vibration and is related to th mutually perpendicular lines. Hence, if the ether vibrates freely a disturbance parallel to one or other of these lines, the period of vibrat will depend on the direction of the disturbance; for, though the effec elasticity is the same for each direction, the effective density, or effec mass to be put in motion, is different: and the ideal force, which acting an ideal particle of unit mass gives a synchronous representative vibrati will have a different relation to the distance for the three directions disturbance, although the ethereal elasticity, both actual and effective assumed to be really identical for all directions.

A fallacy.

17. We are now in a position to recognise the fallacy of a met which has been used for the derivation of Fresnel's wave-surface from properties of an incompressible elastic ether. It is first proved that elastic force evoked by a unit displacement along a line OP, which the radius vector of a certain ellipsoid, if resolved along the direction displacement OP, is $\frac{1}{OP^2}$: that if OP is an axis of a section of the ellips the other component is perpendicular not merely to OP but also to plane of the section: that if the section has the direction of the wa front, the second component is without effect owing to the incompre bility of the ether: that the effective elastic force for unit displacem is thus $\frac{1}{OP^2}$. It is then tacitly assumed that the effective elastic force identical with the above representative force (which is measured by $\frac{4}{}$ times the distance): hence it is inferred that $\frac{4\pi^2 v^2}{\lambda^2} = \frac{1}{OP^2}$. It is wrongly assumed that *the wave-length* λ *is always the same for rays of the s colour transmitted in the same medium*, and that λ in the above relatio thus a constant: whence it is concluded that the velocity varies inversel OP. That the proof is fallacious is clear from the last Article, in which it been shown that in the representative vibration the relation of the ideal f

e ideal distance is independent of the specific properties of the medium depends on the luminous source. Indeed the assumption of the constancy is inconsistent with the conclusion, namely that v differs with the direction bration; it is obvious that the period $\frac{\lambda}{v}$ and the wave-length λ cannot be constant if v be variable: the colour really depends on the *period of tion*, not solely on the wave-length.

esnel himself proceeded in a different way, and assumed a relation ded on the analogy of a line of vibrating ethereal particles to a ating string.

neral, if a plane-polarised ray is transmissible in a given direction, the plane of polarisation can have at most two different directions.

3. We have seen (Art. 8) that if two plane-polarised rays of the wave-length can be transmitted along the same line with the same ty but with different positions of the plane of polarisation, they may be cal in effect with a single plane-polarised ray transmitted along the with the same velocity but with an intermediate position of the plane larisation; the direction of the latter being determined by the ratio e amplitudes of vibration of the component rays: conversely, the of a single ray of given plane of polarisation and simple colour is cal with that of two rays of the same simple colour transmitted along me line with the same velocity, and with their planes of polarisation y assigned positions.

w a plane-polarised ray can be transmitted along the line of inter- n of two planes of physical symmetry of the medium, for the planes mmetry of the plane-polarised ray and the planes of symmetry of edium may be taken to coincide: but the velocity of the ray will d upon the position of the plane of polarisation, if the physical rela- of the medium relative to the two planes of symmetry are different. e latter be the case, as for instance when the line is an axis of sym- of an ortho-rhombic crystal, no ray having a plane of polarisation ue to the symmetral planes of the crystal can be transmitted along r such a ray, if transmissible, would be kinematically equivalent to ays transmitted along the line with the *same* velocity, each having ane of polarisation coincident with a different plane of symmetry; ays can be actually transmitted with these positions of the plane of sation, but that their velocity should be equal is in general phy- y impossible.

exactly the same way it follows that if along any line, whether an

axis of symmetry or not, a plane-polarised ray can be transmitted wit plane of polarisation in two different directions but with different veloc in the two cases, a third position of the plane of polarisation is physi impossible.

The refraction of the medium cannot be higher than double.

19. Hence, for a given direction of transmission in such a mediu plane-polarised ray cannot have more than two different velocities: the medium cannot present more than double refraction; for, accordin the undulatory theory, whatever the nature of the physical change, direction of the refracted ray is dependent upon its velocity.

Degree of the equation of the ray-surface.

20. A diameter of the ray-surface for such a medium will thus in sect the surface in at most four real points, two on each side of the ori and the equation of the ray-surface cannot be of a degree higher than fourth, if it be granted that the above method of proof excludes the e tence of imaginary velocities and imaginary points of intersection of a line with the surface.

In fact, even if there be two imaginary positions of the plane of pol sation for a given real direction of ray-transmission, the imaginary veloc must be in general unequal, since the two planes will be differently rel to the crystal and will thus correspond to different crystalline proper whether real or imaginary. But even if the planes of polarisation imaginary, the *difference* of the imaginary velocities of the two pl polarised rays prevents the resultant effect from being that of a si plane-polarised ray with an imaginary plane of polarisation (Art. 8*b*).

The transmissibility of even a single plane-polarised ray is not a phys necessity; but if one position of a plane of polarisation be possible, ther a second at right angles with the first.

21. We may remark that it is not a physical necessity that a pl polarised ray should be transmissible at all: a plane-polarised ray can be transmitted, for instance, along the morphological axis of a crysta quartz.

As a plane-polarised ray is symmetrical to two planes, the plan polarisation and the transverse plane, it would seem that if the charac of a crystal admit of one symmetral plane of the ray having a gi position, they must admit of the other symmetral plane having the s position: in other words, for a given direction of transmission, if ther

possible position of the plane of polarisation, there is a second at right
es to the first. The same result is later arrived at in another way and
positions of the perpendicular planes are determined (Art. 40*c*).

smission of a ray along an axis of tetragonal or hexagonal symmetry.

2. On the other hand, the morphological axis of a tetragonal or hexa-
l crystal is the intersection of two or more symmetral planes for
h the physical relations are identical: hence along such a line it is
ically possible to transmit two rays having the same velocity and
rent planes of polarisation, and thus having a resultant effect identical
that of a single plane-polarised ray. The amplitudes of the component
being arbitrary may be so adjusted that the equivalent single plane-
rised ray has any plane of polarisation whatever: and it follows that
g the morphological axis of a given tetragonal or hexagonal crystal a ray
be transmitted with any direction of the plane of polarisation, but in
case with the same velocity.

The velocity-factor.

3. The velocity of transmission of a plane-polarised ray of given
r is found to depend on the properties of the medium: since the
tion is in only a single direction, we may assume that the velocity of
mission corresponding to a given direction of vibration depends
on the properties of the medium relative to the direction of the
tion. To avoid confusion of ideas, let the action of the medium in so
as it affects the velocity of a ray of given direction of vibration be
de be due to a *velocity-factor*; the magnitude of the factor depending
ne properties of the medium for the direction of the periodic change
bration.

elocity-factor is necessarily the same for all directions perpendicular to an axis of tetragonal or hexagonal symmetry.

4. We have shown, from principles of mere symmetry of the medium
superposition of changes, without regard to their physical character,
along the morphological axis of a tetragonal or hexagonal crystal a ray
nsmissible with the plane of polarisation in any azimuth whatever,
hat the velocity of transmission of the ray is always the same: hence,
ll directions of vibration perpendicular to the morphological axis of a
al crystal, the velocity-factor has the same magnitude. It follows
the symmetry of the velocity-factor, at any rate for directions of

rectilinear vibration lying in a plane perpendicular to the tetragona hexagonal axis of symmetry, is of a higher order than that of the r phological development.

The corresponding geometrical character is worthy of remark, nam that in a parallelepipedal system of points every plane of the sys passing through an axis of tetragonal or hexagonal symmetry is a plan symmetry for the planes and lines, though not for the points, of the syst the symmetry of the system relative to such a plane being in gen "symmetry of aspect," and not absolute.[1]

Transmission of a ray in a direction lying in a plane of physical symm but oblique to an axis of tetragonal or hexagonal symmetry.

25. Consider the case of a ray transmitted in one of the planes of s metry S of a tetragonal crystal, but in a direction oblique to the morpholo axis. Either plane of symmetry of the plane-polarised ray may be ta to coincide with the plane of symmetry S of the crystal: this is confirme experiment, for these directions of the planes of polarisation of a ray found to be physically possible. But if the plane of polarisation of ray is coincident with the plane of symmetry S of the crystal, and vibration is assumed to be perpendicular to the plane of polarisation, vibration is perpendicular to the morphological axis whatever the posi of the ray in the plane: hence, according to the preceding Article, velocity-factor, and therefore the velocity of the ray, will be the same all ray-directions in this plane, and one curve of intersection of the surface with the plane of symmetry S of the crystal will be a circle. the other hand, if the plane of polarisation of the ray is normal to the p of symmetry S of the crystal, the vibration will be in the same plan symmetry S and in a direction oblique to the morphological axis: physical characters belonging to the direction of the vibration, inclu the velocity-factor, will thus vary with the direction of the ray, and velocity-curve corresponding to those rays of which the plane of pola tion is normal to the symmetral plane S of the crystal will not be circu the curve will be symmetrical, however, both to the morphological and a line perpendicular to it, for they are directions with respect which all the characters of the crystal are symmetrical. Further, second curve will touch the first at its points of intersection with morphological axis: for the two directions perpendicular to that line, lying respectively in and perpendicular to the plane of symmetry S, ar

[1] H. J. S. Smith; *Philosophical Magazine*, 1877, ser. 5, vol. 4, p. 18.

othesis similar in all their relations, and correspond therefore to the e velocity-factor; hence both curves meet on the morphological axis, therefore touch each other, for the morphological axis divides each e symmetrically.

ut since the equation of the ray-surface has been shown to be of a ee not higher than the fourth, and the equation of one curve of rsection, a circle, is of the second degree, that of the other curve likewise be of the second degree, and therefore represent an ellipse — the curve is closed and has unequal diameters. This result agrees the experimental discovery made by Huygens.

smission of rays along the axes of symmetry of an ortho-rhombic crystal.

6. Take next the case of an ortho-rhombic crystal. In the first e, as shown in Art. 18, a ray can be transmitted along any of the axes mmetry, and have its plane of polarisation coincident with either of symmetral planes of the crystal which intersect therein. The three of symmetry being independent of each other in all their physical ions, the velocity-factors will be independent; and vibrations paral- o the several axes will thus in general correspond to different veloci- of transmission. Let the velocity corresponding to an axis OX, OY, OZ, *considered as a direction of vibration*, be denoted by a, b, or c ectively: then two rays are transmissible along OX with velo- s b and c, and planes of polarisation normal to OY and OZ respec- y: two rays are transmissible along OY with velocities c and a, and es of polarisation normal to OZ and OX respectively: two rays are smissible along OZ with velocities a and b, and planes of polarisation mal to OX and OY respectively.

unsmission of rays in a symmetral plane of an ortho-rhombic crystal.

7. Again, as far as directions lying in the plane of symmetry OXZ concerned, there is no essential difference between an ortho-rhombic a tetragonal crystal, if OZ is the morphological axis of the latter. essential difference between two such crystals is that in one of (the ortho-rhombic) the third axis of symmetry OY is independent X in its physical relations, and in the other (the tetragonal) is iden- therewith. Hence we may infer that in the symmetral plane OXZ of rtho-rhombic crystal, a ray is transmissible in any direction with its e of polarisation either coincident with or perpendicular to that plane. , as in the case of a tetragonal crystal, the intersection of the ray-

surface with the symmetral plane will be a circle and a concentric ellip
but, in the ortho-rhombic crystal, the circle and ellipse will be independ
of each other in magnitude, since the velocity-factor for the direction
vibration OY is independent of that for the direction of vibration OX.

Intersections of the ray-surface with the symmetral planes of an ortho-rhombic crystal.

28. The intersections of the ray-surface with the axial planes O
OZX, OXY of an ortho-rhombic crystal will thus be given by the follo
equations:—

$$(y^2+z^2-a^2)(b^2y^2+c^2z^2-b^2c^2)=0,$$
$$(z^2+x^2-b^2)(c^2z^2+a^2x^2-c^2a^2)=0,$$
$$(x^2+y^2-c^2)(a^2x^2+b^2y^2-a^2b^2)=0.$$

General equation of the ray-surface for an ortho-rhombic crystal.

29. The equation of the ray-surface itself must be of the form

$$(y^2+z^2-a^2)(b^2y^2+c^2z^2-b^2c^2)+x\,\phi(xyz)=0,$$

since it reduces to the first expression when x is made zero. But
cording to Art. **20** the quantity $x\,\phi(xyz)$ cannot consist of terms of deg
higher than the fourth: further, the surface being symmetrical to the a
planes, its equation can only involve even powers of $x\,y\,z$: hence the
terms which can enter the expression $x\,\phi(xyz)$ are x^4, x^2z^2, x^2y^2 and x^2.

The general equation is thus of the form

$$(y^2+z^2-a^2)(b^2y^2+c^2z^2-b^2c^2)+Ax^4+Bx^2z^2+Cx^2y^2+Dx^2=0;$$

or, multiplying out,

$$Ax^4+b^2y^4+c^2z^4+(b^2+c^2)y^2z^2+Bx^2z^2+Cx^2y^2+Dx^2-b^2(c^2+a^2)y^2$$
$$-c^2(a^2+b^2)z^2+a^2b^2c^2=0.$$

Also, it is evident from the equations of the curves of intersection
the three axial planes that xyz and abc are simultaneously cyclically i
changeable (Art. **28**); hence

$$A=a^2;\ B=c^2+a^2;\ C=a^2+b^2;\ D=-a^2(b^2+c^2).$$

Substituting these values, the equation becomes

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(c^2+a^2)y^2-c^2(a^2+b^2)z^2+a^2b^2c^2$$

or, multiplying by r^2,

$$r^4(a^2x^2+b^2y^2+c^2z^2)-r^2\{a^2(b^2+c^2)x^2+b^2(c^2+a^2)y^2+c^2(a^2+b^2)z^2\}+a^2b^2c^2(x^2+y^2+z^2$$

or $a^2x^2(r^2-b^2)(r^2-c^2)+b^2y^2(r^2-c^2)(r^2-a^2)+c^2z^2(r^2-a^2)(r^2-b^2)=$

$$\text{or}\quad \frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0:$$

which is Fresnel's equation of the ray-surface.

The ray-surface for a mono-symmetric or anorthic crystal.

0. (*a*) Next consider the case of a crystal which admits of the trans-ion of a ray of plane-polarised light in any direction, but presents only ngle plane of geometrical and physical symmetry, and thus belongs to mono-symmetric system: let the normal of the plane of symmetry *OY*.

ince, from a purely geometrical point of view, a mono-symmetric tal may be regarded as a homographic transformation of an ortho-nbic crystal, it first suggests itself that the ray-surface for a mono-metric crystal may be such as would result from a corresponding trans-ation of the ray-surface for an ortho-rhombic crystal. That the analogy perfect, however, is evident from the fact that there is no corresponding rtion of the planes of polarisation; whatever the direction of ray-smission within the mono-symmetric crystal, the planes of polarisa-of the two transmissible rays are perpendicular to each other (Art.

As in Art. **27**, any ray whatever lying in the plane of symmetry can that plane for either its plane of polarisation or its transverse plane: e, exactly in the same way as before, it follows that the plane of symmetry sects the ray-surface in two curves, the one a circle, the other a con-ic ellipse: the former corresponding to the rays which have the metral plane for the plane of polarisation, the latter to the rays for which plane of symmetry is the transverse plane. If *OX OZ* be the axes of ellipse, a ray transmitted along *OX* will thus have its representative ations parallel to either *OY* or *OZ*; and a ray transmitted along *OZ* have its vibrations parallel to either *OY* or *OX*.

A ray transmissible along the line *OY* can have its plane of polari-n in only one or other of two directions of which the normals are endicular both to each other and to the line *OY* (Art. **21**).

Since the elliptic and circular sections of the ray-surface made by plane *XOZ* are both of them symmetrical to the lines *OX OZ*, while plane *XOZ* is a plane of general physical symmetry of the crystal, and normal *OY* is an axis of general symmetry of diagonal type, we may onably assume that *for this particular property* (so long as there is no ation of colour or temperature) the planes *YOX YOZ* are themselves es of symmetry of the crystal; in which case, the lines *OX OZ* will be directions of vibration of the two rays transmissible along the axis *OY*.

For the given colour and temperature, the circumstances are tical, for this particular property, with those of an ortho-rhombic tal having *OX OY OZ* for axes of symmetry: and the ray-surface

will thus for a mono-symmetric crystal have the same general form as an ortho-rhombic one.

(*b*) The general form of the ray-surface, being quite unaffected by degradation of the symmetry from the ortho-rhombic to the mono-symmetric type, is clearly independent of the type of symmetry altogether the general form will therefore be the same even for an anorthic crystal.

The difference in the type of symmetry thus affects, not the general form of the ray-surface, but only the constancy of the directions and relative lengths of the axes of the surface for different colours and temperatures. An axis of general symmetry of the crystal is necessarily an axis of symmetry of the ray-surface whatever the colour of the light or the temperature of the crystal (p. 297).

The form of the ray-surface is independent of the physical character of the periodic change.

31. The rigorous accuracy of the form assigned to the ray-surface by Fresnel is thus a necessary consequence of the general features of perpendicularly transverse undulations, independently of the physical character of the change.

And although, as in the case of an incompressible elastic ether with effective rigidity dependent upon the direction of vibration, the same form of ray-surface may result notwithstanding the obliquity of the transverse vibration, this is not generally true. The form of the ray-surface which follows, for example, from a version of the elastic theory of double refraction suggested by Rankine and further developed by Lord Rayleigh is different from that of Fresnel, and only gives the latter as a first approximation. That version, according to which the ether is incompressible but has an effective density dependent on the direction of vibration, involves the general obliquity of the latter to the direction of transmission.[1]

In fact, whatever the degree of symmetry of the characters of a plane polarised ray as transmitted within a medium, the above form of ray-surface will result from any hypothesis which has for necessary consequence that if one plane-polarised ray is transmissible in a given direction a second plane-polarised ray is transmissible in the same direction with a different velocity and has its plane of polarisation perpendicular to that of the first.

It may be remarked that in the above reasoning no assumption as to the molecular constitution of the ether has been necessary.

[1] *Philosophical Magazine;* 1851, ser. 4, vol. 1, p. 441; 1888, ser. 5, vol. 26, pp. 525,

Resilience.

32. It was explained in Art. 14 that the period of the change at any point a ray of light depends in general on the period of the change at the minous source, and not on the specific properties of the medium; but latter may conceivably affect some or all of the remaining characters of ray, namely, amplitude and direction of vibration, velocity and direc- n of transmission through the medium. The property by virtue of ch a periodic change of any kind is transmissible through a medium y be denoted by the general term *resilience*: we may imagine that a urbance at any point of the medium evokes an opposing resilience of ch the magnitude increases with the amount of the disturbance. ical resilience, in so far as it affects only the velocity of transmission a periodic change having a given direction, is identical with the city-factor for that direction, mentioned in Art. 23. When the periodic nge is a vibratory motion such as follows the removal of a compressing distorting force, resilience is identical with elasticity of volume or re.

From this point of view, the periodic change at any point of a of plane-polarised light may be treated as a resultant effect of two es; the one an *initiatory* linear force periodic in its variations, and ing a period identical with that of the luminous source; the other a *ular* force or a resilience, evoked by the disturbance produced by the iatory force.

Free and forced vibrations.

33. The periodic change at a point of a ray of light is a *forced* vibra- , resulting from the continued action of the luminous source: it differs a *free* vibration, such as would be produced by resilience alone if luminous source were removed while the medium is in a state of urbance.

A simple case of free vibration.

34. In the simplest possible case of free vibration of a character of a dium, we may imagine that the disturbance at the point is of such a d that at any instant it can be represented by the length and direc- of a straight line y drawn from an ideal particle of unit mass to the nt, and that the resilience of the medium can be represented by an l force acting in the line of disturbance, tending to diminish the dis- bance, and proportional in magnitude to the disturbance itself, the portion being independent of direction: such a medium may be said e *isotropically* resilient for the given character.

As in Art. 13 we may write

$$\frac{d^2y}{dt^2} = -f^2y,$$

where f is independent of the time and depends on the properties of medium.

A solution of this equation is $y = B\sin(ft+\beta)$, where B and β constants: the expression represents a vibration of which the perio $\frac{2\pi}{f}$, since any value of y recurs when t is increased by an integral multi of that quantity.

As already pointed out, such a mode of representation is still possi even when the actual change is an oscillatory rotation of an ethe particle (Art. 13).

A simple case of forced vibration.

35. But suppose that in the above medium the vibration at the poi not free but forced, and that the initiatory force is a periodic one relate the time in the same way as the disturbance at a point of a plane-polar ray of simple colour; the initiatory force can in such case be represented an expression of the form $S\sin st$, where S and s are constants, and the la depends only on the period of vibration of the luminous source. As fore, the ideal resultant force acting on the ideal particle of unit ma $\frac{d^2y}{dt^2}$, and is due to the superposition of the initiatory force $S\sin st$ and resilience $-f^2y$: hence $\frac{d^2y}{dt^2} = S\sin st - f^2y$.

It is easily seen that $y = B\sin st$ is a solution of this differential equat for differentiating, we get first $\frac{dy}{dt} = Bs\cos st$, and next $\frac{d^2y}{dt^2} = -Bs^2\sin$ substituting in the above equation, and dividing by $\sin st$, we $B = \frac{S}{f^2-s^2}$: and thus $y = \frac{S}{f^2-s^2}\sin st$. Hence the resilience af merely the amplitude, not the period or general character of the vibra at the point.

The resilience being $-f^2y$ has likewise the same period as the initia force.

Transmission of a simple forced vibration in an isotropically resi medium.

36. If a luminous source is in a state of periodic vibration represe

matically by the linear motion of a particle attracted to an origin by a e proportional to the distance, and is surrounded by a medium such the resilience is represented by a force acting in the line of disturb- e and proportional to it in magnitude, the changes transmitted through medium along a given direction perpendicular to that of the vibra- may thus be expected to be always in the same plane and have same period; no resilient force oblique to the plane containing the ction of ray-transmission and the direction of vibration of the luminous rce is evoked by the disturbance: in any direction of an isotropically lient medium, a plane-polarised ray, if transmissible at all, may thus transmitted with any azimuth of plane of polarisation whatever.

more general case of free vibration of an œolotropically resilient medium.

37. As a more general case, we may imagine that in a crystalline lium there are three directions, not co-planar, inclined obliquely or rpendicularly to each other, for each of which a disturbance evokes a ilience which in its effects is represented by an ideal force, contrary and portional to the disturbance, acting on an ideal particle of unit mass; relation of the ideal representative force to the distance of an ideal cted particle of unit mass being, however, like most other physical racters, different for the three directions: the latter may be termed *of optical resilience.*

That such a representation is possible, even in an elastic ether of which elasticity is the same in all directions, has already been pointed out in **16**: for if the effective density depends on the direction of vibration, period of a free vibration will also vary with the direction, since ough the real accelerative force has the same constant relation to the ance it will have a different effective mass to keep in motion.

When it is desirable to emphasise the fact that the resilient force under sideration is the ideal force which would produce an analogous to-and- motion of a particle of unit mass and not the statical force necessary to maintenance of a given state of disturbance, we may conveniently dis- guish it as *vibrational resilience.* If the three constants of vibrational ilience be respectively $e^2\ f^2\ g^2$, and $x\ y\ z$ be the distances which re- sent the disturbances parallel to the respective axes at any time t, we e for a free vibration due to a disturbance along each of the axes

$$\frac{d^2x}{dt^2} = -e^2x\,;\ \frac{d^2y}{dt^2} = -f^2y\,;\ \frac{d^2z}{dt^2} = -g^2z:$$

ence, in the same way as before,

$$x = A\sin(et + \alpha)\,;\ y = B\sin(ft + \beta)\,;\ z = C\sin(gt + \gamma).$$

According to the principle of superposition of changes, if the direct of the initial disturbance at the point is inclined to the three axes resilience, the initial disturbance may be resolved along those directio and the resultant free vibration is such as would result from the co position of the free vibrations corresponding to the several axial dir tions. Hence, if the vibration is free, the disturbance at a given inst is determined by the above triad of equations.

Since the ratios $x : y : z$ depend on the time, the motion of the rep sentative particle is not in a straight line passing through the ori The particle in fact describes a curve in three dimensions, and ne passes twice through the same position unless the ratios $e : f : g$ are c mensurable.

The quantities $e^2 f^2 g^2$ may be conveniently termed *coefficients of opt vibrational resilience*: and the medium may be said to be *æolotropic* resilient. The coefficients of vibrational resilience are independent and t for the same ray, but even with the same medium may conceiva be different for different rays, and thus vary with the period of the char or in other words, with the colour of the light.

In Art. 42 it is pointed out that obliquity of mutual inclination of axes of optical resilience is not met with even in mono-symmetric anorthic crystals.

A more general case of forced vibration of an æolotropically resilient medi

38. Consider next a forced vibration of a crystalline medium hav three dissimilar oblique or rectangular axes of vibrational resilience before : assume that the initiatory force at any point of a ray may again represented by an expression of the form $S \sin st$, where s is a const depending on the period of the change at the luminous source.

If OP be any line passing through an origin O, and OL OM lengths measured along the axes of resilience, be edges of a parallelepi of which OP is a diagonal, whatever the length OP we have

$$OL = \lambda \cdot OP \quad OM = \mu \cdot OP \quad ON = \nu \cdot OP,$$

where λ μ ν are constants for a given direction of OP.

From the principle of superposition, it follows that the initiatory f $S \sin st$ acting in the line OP can be resolved into three initiatory fo $\lambda S \sin st$ $\mu S \sin st$ $\nu S \sin st$, acting along the axes of x y z respectively.

In exactly the same way as before we have the following expressi for the several vibrations parallel to the respective axes:—

$$x = \frac{\lambda S}{e^2 - s^2} \sin st; \quad y = \frac{\mu S}{f^2 - s^2} \sin st; \quad z = \frac{\nu S}{g^2 - s^2} \sin st;$$

ere the quantities $\lambda\ \mu\ \nu\ e^2 f^2 g^2\ S$ and s are all independent of the time. Hence the ratios $x : y : z$ are also independent of the time, and the resentative particle vibrates in a straight line through the origin.

The period of the resultant vibration is identical with that of the iatory force, but the direction of the vibration is different. If $\lambda'\ \mu'\ \nu'$ ermine the direction of the resultant vibration,

$$\lambda' : \mu' : \nu' = x : y : z = \frac{\lambda}{e^2 - s^2} : \frac{\mu}{f^2 - s^2} : \frac{\nu}{g^2 - s^2}.$$

urther, the axial components of the representative resilient force being $x,\ -f^2 y,\ -g^2 z$, or

$$-\frac{e^2 \lambda S}{e^2 - s^2} \sin st, \quad -\frac{f^2 \mu S}{f^2 - s^2} \sin st, \quad -\frac{g^2 \nu S}{g^2 - s^2} \sin st,$$

resultant resilient force will have a direction determined by the ratios

$$\frac{e^2 \lambda}{e^2 - s^2} : \frac{f^2 \mu}{f^2 - s^2} : \frac{g^2 \nu}{g^2 - s^2}$$

ence the resultant resilient force always acts in the same direction ughout the vibration, but it is inclined to the direction $\lambda\ \mu\ \nu$ of the iatory force and also to the line of vibration $\lambda' \mu' \nu'$, both of which pass ugh the origin: further the resultant resilient force has the same od as the initiatory force.

nission of a simple forced vibration in an œolotropically resilient medium.

9. In such a medium, therefore, an initiatory linear periodic force ng a direction inclined to an axis of resilience and acting at a given t gives rise at that point to a linear periodic vibration in a direction ced to the initiatory force, and to a resilience of which the resultant is represented by a periodic force acting on the ideal particle in a and constant direction not passing through the given point. Since periodic change is transmitted through the medium by virtue of the ence, and action is always equal and contrary to reaction, we should expect that along any line of transmission the direction of the periodic ge will in general vary from point to point of the ray; and that the mitted periodic change can only be in a direction lying always in the plane, if the plane containing the initiatory force and the direction of mission likewise contains the direction of the resilient force, and fore also the direction of representative vibration.

nsider, for example, the case of a ray transmissible along an axis OX ortho-rhombic crystal: from the symmetry it follows that a plane-

polarised ray transmitted along OX must have its vibrations parallel one or other of the dissimilar axes OY OZ, and that for a ray of giv simple colour the velocity of transmission will depend on the direction vibration. If, however, the initiatory force at the initial point of the r though perpendicular to OX, is oblique to the axes OY OZ, it may be solved into two forces, parallel to OY OZ respectively, and each may regarded as originating a simple plane-polarised ray: the motion of representative point will be the resultant of the motions belonging to e ray, and will thus be continually changing its direction as the disturba is transmitted along OX.

Case of an ortho-rhombic crystal.

40. (*a.*) *Direction of the resultant vibrational resilience for a g disturbance.* For simplicity, let the crystalline medium present th mutually perpendicular but dissimilar symmetral planes, and thus bel to the ortho-rhombic system: the axes of resilience necessarily coinc with the crystallographic axes, the lines of intersection of the symme planes. Let $X\,Y\,Z$ be the components, parallel to the axes of co-ordina of the representative resilient force corresponding to a disturbance defi by the co-ordinates $x'\ y'\ z'$: then

$$X=-e^2x';\ Y=-f^2y';\ Z=-g^2z'.$$

The direction-cosines of the resultant resilience F are in the ra

$$X:\ Y:\ Z:: \text{or } e^2x' : f^2y' : g^2z'.$$

But if an ellipsoid $e^2x^2+f^2y^2+g^2z^2=1$ be of such dimensions tha passes through the point P $(x'y'z')$, the direction-cosines of PG normal of the ellipsoid at the point $x'y'z'$ are likewise in the ra $e^2x' : f^2y' : g^2z'$ (Fig. 19).

Hence the resultant resilient force due to a disturbance OP act the direction PG of the normal of the ellipsoid $e^2x^2+f^2y^2+g^2z^2=$ the point P $(x'y'z')$ lying on its surface.

(*b.*) *Direction of transmission of a plane-polarised ray of which the d tion of the plane of polarisation is given.*

If O be the centre of the ellipsoid and OP the representative directio vibration, the initiatory force must also lie in the plane OPG which tains the direction of vibration and the secondary force: further, direction of transmission must lie in the same plane (Art. **39**). vibration being always perpendicularly transversal to the directio transmission, the ray Or corresponding to the vibration OP is thu the plane OPG and perpendicular to OP (Fig. 19).

(*c.*) *The planes of polarisation for a given direction of ray are per*

ılar to each other, and have directions which can be defined by means of ellipsoid.

t has already been proved (Section IV, Art. **24**) that if OP be a central ius vector of an ellipsoid, and PG the normal of the ellipsoid at P, the OP is an axis of the section of the ellipsoid by a plane through OP pendicular to the plane OPG; for a given direction of ray Or re are thus two possible directions of vibration OP_1 OP_2, which can transmitted without change of plane, and they are the axes of the ion of the ellipsoid by a plane to which the ray Or is normal. Hence planes of polarisation corresponding to a given direction of ray are pendicular to each other and are determined by the above geometrical struction.

d.) *Magnitudes of the total and effective vibrational resilience for a given urbance.*

F be the resultant resilience, $F^2 = X^2 + Y^2 + Z^2 = e^4x'^2 + f^4y'^2 + g^4z'^2$. ut if OM (Fig. 19) be the central normal to the tangent plane at $x'y'z'$),

$$\frac{1}{OM^2} = e^4x'^2 + f^4y'^2 + g^4z'^2. \quad \text{(Sec. IV, Art. 4).}$$

ence the resultant resilience F is measured by $\frac{1}{OM}$.

he resilience being in the direction PG, and the actual vibration in irection PO, the effective resilience is $F \cos OPG$

$$= F\frac{OM}{OP} = \frac{1}{OP}.$$

his corresponds to a disturbance of magnitude OP: hence the effective ience for a unit disturbance in the direction OP is $\frac{1}{OP^2}$.

.) *Relation between the effective vibrational resilience and the velocity ransmission.*

a the development of his theory of Double Refraction, Fresnel was pelled to make an assumption as to the relation between the effective ic force and the velocity of normal propagation of the corresponding , and supported his assumption by reference to the analogy of a ting string.

. the preceding Articles, all the forces are purely representative, and assumptions and reasoning founded thereon are really independent of physical character of the change. But it is clear that the velocity of mission must depend on the physical character of the periodic change, hat it is impossible to proceed farther and deduce the absolute velocity

of transmission without some assumption involving the nature of the ch and the constitution of the ether. All that we have been able to sug hitherto is that the velocity is in some way dependent on the charac of the medium relative to the direction of the vibration, and t characters have been collectively expressed by the term velocity-fac in other words, it was suggested that the velocities of the two rays tr missible in the direction Or are determined by some function of directions of vibration OP_1 OP_2, and thus by some function of the len OP_1 OP_2, for the length of a radius vector of an ellipsoid is determ by the direction. But we have also shown (Art. 29) that, without assumption as to the real nature of the change, it is possible to deter the velocities $r_1 r_2$, of the two rays transmissible in a given directio terms of $a\ b\ c$, the velocities corresponding to vibrations in the direct of the principal axes: hence the velocity r of transmission along O necessarily so related to OP, an axis of the section of the ellip $e^2x^2+f^2y^2+g^2z^2=1$ by the plane perpendicular to Or, that the equatio

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0$$

represents the ray-surface. There is only one relation between r and which leads to this form of ray-surface, namely $r=OP$: we are compelled to infer that the velocity of transmission of a ray is directly portional to that radius vector OP of the above ellipsoid $e^2x^2+f^2y^2+g^2z^2$ which has the same direction as the vibrations of the ray: from Art. it follows that the same relation is expressed by the statement that the velocity is *inversely* proportional to the square root of the effective resili for unit disturbance in the direction of vibration.

It follows from the above relation that:—

If a line Or is perpendicular to a central section of the ellip $e^2x^2+f^2y^2+g^2z^2=1$, and OP_1 OP_2 are the axes of the section, a pl polarised ray can be transmitted along Or, having OP_1 or OP_2 fo normal of its plane of polarisation and a velocity of transmission meas by OP_1 or OP_2 respectively.

That this relation is consistent with the form of the ray-surface ar at in Art. 29 may be proved as follows:—

(*f.*) *Transformation of the above construction.*

From O draw OM perpendicular to the plane which touches ellipsoid $e^2x^2+f^2y^2+g^2z^2=1$ at the point P, and let M be the fo the perpendicular: produce OM to R, making $OR\cdot OM=1$.

(1.) First—find the locus of the points R when P takes all posi on the ellipsoid $e^2x^2+f^2y^2+g^2z^2=1$.

et $x'y'z'$ be the co-ordinates of P (Fig. 19); the tangent plane at P is
$+f^2y'y+g^2z'z=1$: hence, as in Art. 40*d*,

$$\frac{1}{OM^2}=e^4x'^2+f^4y'^2+g^4z'^2.$$

$\xi\ \eta\ \zeta$ be the co-ordinates of the point R, we have $\frac{\xi}{e^2x'}=\frac{\eta}{f^2y'}=\frac{\zeta}{g^2z'}$;
by construction, the line OMR is perpendicular to the tangent plane
$+f^2y'y+g^2z'z=1$.

ach of these fractions is equal to $\frac{\sqrt{(\xi^2+\eta^2+\zeta^2)}}{\sqrt{(e^4x'^2+f^4y'^2+g^4z'^2)}}$; that is $OR\cdot OM$,
nity.

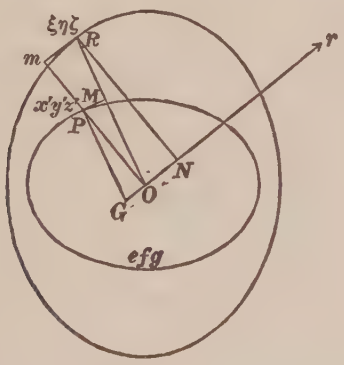

Fig. 19.

e thus have $\frac{\xi}{e}=ex';\ \frac{\eta}{f}=fy';\ \frac{\zeta}{g}=gz'$:

ce, since $e^2x'^2+f^2y'^2+g^2z'^2=1$,

lows that $\frac{\xi^2}{e^2}+\frac{\eta^2}{f^2}+\frac{\zeta^2}{g^2}=1.$

is is the equation of the locus of the points R, and represents an
oid with the same symmetral planes as $e^2x^2+f^2y^2+g^2z^2=1$, but with
axes $e\ f\ g$ instead of $\frac{1}{e}\ \frac{1}{f}\ \frac{1}{g}$.

). The equation of the plane which touches the ellipsoid $\frac{\xi^2}{e^2}+\frac{\eta^2}{f^2}+\frac{\zeta^2}{g^2}=1$
e point $R\ (\xi\ \eta\ \zeta)$ is

$$\frac{x\xi}{e^2}+\frac{y\eta}{f^2}+\frac{z\zeta}{g^2}=1.$$

If Om be the perpendicular on this plane from the origin, the direct cosines of Om have the ratios $\frac{\xi}{e^2} : \frac{\eta}{f^2} : \frac{\zeta}{g^2}$, or $x' : y' : z'$; hence the line passes through the point P.

RN, the normal of the ellipsoid $\frac{\xi^2}{e^2}+\frac{\eta^2}{f^2}+\frac{\zeta^2}{g^2}=1$ at R, is parallel Om, and will thus intersect the line Or which lies in the plane ORP N be the point of intersection, $RN = Om$.

(3.) In the same way as before, since Om is the perpendicular fr the origin to the plane $\frac{x\xi}{e^2}+\frac{y\eta}{f^2}+\frac{z\zeta}{g^2}=1$,

$$\frac{1}{RN^2}=\frac{1}{Om^2}=\frac{\xi^2}{e^4}+\frac{\eta^2}{f^4}+\frac{\zeta^2}{g^4}=x'^2+y'^2+z'^2=OP^2.$$

The plane ORN is thus identical with the plane OPG; the nor RN to the ellipsoid at the point R has the same direction as OP a represents the direction of the vibration: $\frac{1}{RN}$ is equal to OP and the fore measures the velocity of transmission.

Hence the relation given above in Art. **40***e* is equivalent to t following:—If Or is the direction of transmission of a ray, the direct of the vibration, or the normal of the plane of polarisation, is nor to the line Or and also to the ellipsoid $\frac{x^2}{e^2}+\frac{y^2}{f^2}+\frac{z^2}{g^2}=1$; its veloc is measured by the *inverse* of the length of this normal intercepted b tween the ray and the ellipsoid: further the normal of the ellipsoid is p pendicular to the plane of polarisation of the corresponding ray.

It has been proved in Section IV that the ray-surface which follo from this relation is

$$\frac{\frac{1}{e^2}x^2}{r^2-\frac{1}{e^2}}+\frac{\frac{1}{f^2}y^2}{r^2-\frac{1}{f^2}}+\frac{\frac{1}{g^2}z^2}{r^2-\frac{1}{g^2}}=0;$$

it has also been proved (Art. **29**), without any assumption as to the n nature of the periodic change, that the equation of the ray-surface is

$$\frac{a^2x^2}{r^2-a^2}+\frac{b^2y^2}{r^2-b^2}+\frac{c^2z^2}{r^2-c^2}=0:$$

the results are consistent with each other if $e=\frac{1}{a}$, $f=\frac{1}{b}$, $g=\frac{1}{c}$;

oid $\frac{x^2}{e^2}+\frac{y^2}{f^2}+\frac{z^2}{g^2}=1$ is identical with the optical indicatrix

$$a^2x^2+b^2y^2+c^2z^2=1.$$

Comparison with Fresnel's elastic forces.

. (a.) If $a\ b\ c$ are the velocities of transmission of those rays of n the vibrations are parallel to the axes of $x\ y\ z$ respectively, Fres- method of derivation requires the elastic forces evoked by unit dis- ments along the axes to be taken as $a^2\ b^2\ c^2$ respectively: according e above method, the ideal forces, which by their action on an ideal cle of unit mass would produce vibrations synchronous with those of nedium, will be measured by $e^2f^2g^2$ or $\frac{1}{a^2}\frac{1}{b^2}\frac{1}{c^2}$ for unit displacements of deal particle along those directions.

.) In Fresnel's method, the evoked elastic force normal to the direction ration of a real particle of ether is regarded as of no effect owing to the mpressibility of the medium: in the above method no assumption is as to the compressibility or incompressibility of the medium, but component of the representative resilient force which is normal to rection of vibration of an ideal particle is regarded as balanced by ual component of the representative initiatory force at the same of the ray.

Case of a mono-symmetric or anorthic crystal.

2. If the medium could present three dissimilar axes of optical resi- obliquely inclined to each other, it would follow as before that the components of the resilient force, corresponding to a disturbance ed by the co-ordinates $x'y'z'$, would be

$$X=-e^2x';\ Y=-f^2y';\ Z=-g^2z':$$

he resultant resilient force F would no longer act along the normal e ellipsoid $e^2x^2\mp f^2y^2+g^2z^2=1$, and the planes of polarisation of the rays missible along a given direction would no longer be at right angles to other.

such a character is not presented by any crystal which has been ined, we may infer that in all crystalline media the axes of optical ence for a given colour and temperature are always mutually perpen- ar, and that the symmetry of the crystal merely affects the directions e triads of perpendicular lines and the ratios of the corresponding co- ents of optical resilience.

An unsatisfactory variation of Fresnel's method.

3. At first sight it would seem that the following would be a satisfac-

tory and more simple mode of altering Fresnel's assumptions and reas
ing so as to accord with the recent conclusion that the vibration is pa
lel, not to the radius vector RO, but to the normal RN of the ellips
$a^2x^2+b^2y^2+c^2z^2=1$.

Let $-a^2\xi, -b^2\eta, -c^2\zeta$, be the resolved axial components of the ela
force on a particle, due to a displacement from the centre O to the poin
$(\xi\ \eta\ \zeta)$ lying on the surface of the ellipsoid $a^2x^2+b^2y^2+c^2z^2=1$. It n
be shown as above that the elastic force acting on the particle when a
is directed along the normal RN and measured by $\frac{1}{RN}$, N being as us
the point of intersection of the normal with the ray: hence the ela
force *for unit displacement measured perpendicularly to the ray*,
thus parallel to the direction of vibration, is $\frac{1}{RN^2}$.

Hence if the particle were set free after having been displaced to
point R and no other force than the evoked elasticity were acting upo
the *initial* motion would be along the normal RN under the action o
force which is measured by $\frac{1}{RN^2}$ times the distance from the ray. Assu
ing that the vibration is actually and permanently perpendicular to the ra
periodic constraining force is requisite to maintain the isochronous chara
of the motion: if it were possible that the constraining force and the evo
elasticity could together be always measured by $\frac{1}{RN^2}$ times the dista
of the particle from the direction of the ray, we might infer by anal
with the case of sound that the velocity of transmission would
measured by $\frac{1}{RN}$, for $\frac{1}{RN}$ is the square root of the effective elastic f
due to a unit displacement in the direction of vibration.

It will be found, however, that such a motion cannot actually take pl
the particle will only vibrate rectilinearly if the path passes through
origin. If the particle is not moving in a line through the origin,
evoked elastic force will be constantly changing direction; for at
instant it acts parallel to the normals of the ellipsoid $a^2x^2+b^2y^2+c^2z^2=$
the points where a line joining the particle to the centre O meets the surf
Further the initiatory periodic constraining force is zero, not when
particle is at its position of maximum displacement, but when it is i
position of no disturbance.

The transmission of elliptically or circularly polarised rays.

4. We have seen above (Art. 8*a*) that the simultaneous transmission
wo plane-polarised rays of the same simple colour along the same line
the same velocity, but with different directions of planes of polarisa-
, has for general result an elliptically polarised ray of the same
ple colour transmitted with the same velocity: further, the right-hand
eft-hand character of the motion of the representative point round the
se depends only on the relation of the phases of the component rays.
ce, in general, an elliptically polarised ray, or, its special case, a circu-
polarised ray, can be transmitted in any direction within a cubic
tal, or along the morphological axis of a tetragonal or hexagonal crys-
and its velocity is independent of its right-hand or left-hand cha-
er.

the velocity of transmission of a plane-polarised ray along a given
ction within a crystal is dependent on the azimuth of the plane of
risation, we have seen (Art. 8*b*) that an elliptically or circularly
rised ray cannot result from the composition of two plane-polarised
transmitted along that direction.

he transmission of a circularly polarised ray, however, may be possible
when that of a single plane-polarised ray is not so: for instance, a
t-hand or a left-hand circularly polarised ray, but not a plane-polarised
can be transmitted along the morphological axis of a crystal of
rtz. In such case, the velocities of transmission of a right-hand and
ft-hand circularly polarised ray of the same simple colour are neces-
y different: for it will be found on calculation that a right-hand and
t-hand circularly polarised ray *transmitted with the same velocity*, if
posed, are kinematically identical with a plane-polarised ray, the
uth of the plane of polarisation of which depends solely on the relative
es of the component rays; but according to hypothesis a plane-
rised ray is incapable of transmission. Such a line will be an axis of
al symmetry, but cannot lie in a plane of general symmetry; for
metry to the plane would require a right-hand and a left-hand ray to be
smissible with the same velocity.

fact, if a right and left circular motion of the same radius and
d are simultaneously impressed on the same particle, the resultant
on is a vibration along that diameter of the circle to which the two
lar motions are symmetrical, namely, the diameter passing through
wo positions of the particle which are identical for the component
ons. If the two circular motions are transmitted through the
ium with the same velocity, their relative phases, and thus the direc-

tion of the line of resultant vibration, will be the same at all points of t resultant ray: if they are transmitted with unequal velocities, the li of resultant vibration will have different azimuths for different points the ray, and the change of azimuth will be proportional to the distar between the given points. Hence it follows that if a plane-polarised r be incident normally on a plate cut perpendicularly to the morphologi axis of a crystal of quartz, the ray will not be in a state of plane-polari tion within the plate, though it will be so after emergence: the planes polarisation of the incident and emergent rays will be inclined to each ot at an angle which is proportional to the thickness of the plate.

Summary.

1. Fresnel's hypothesis—that light consists in the vibratory motio an incompressible elastic ether—being untenable, should be abandoned an educational instrument.

2. The later hypothesis—that light consists in the vibratory motio a compressible elastic ether, of which the elasticity (of volume and figu is the same for all bodies and for all directions in the same body, an which the effective density in bi-refractive media is dependent on the di tion of the vibratory motion—satisfactorily accounts for most of the kn optical laws: hence such terms as "axes of optical elasticity," wh relate to variation of elasticity, must be discontinued.

3. Even this more satisfactory hypothesis may only be an approxin mechanical analogy, and may eventually be found to be inconsistent v experiment in some of its optical results; hence it cannot be satisfacto used as the *basis* of a correlation of optical characters for the studen crystals; in fact, though it appears to be fully established that elec magnetic waves and light-waves differ only in length, an electro-magn disturbance seems to be inexplicable as mere vibratory motion o elastic body.

4. On the other hand, the accuracy of Huygens' construction is no far confirmed by experiment that it doubtless expresses a Law of Nat

5. This being the case, it is easily seen that the velocity and pola tion of each of the two rays transmissible in a given direction in a uni crystal can be simply expressed by means of the spheroid alone:—

If R be a point on the spheroid, O the centre, RN the normal, N line intersecting the normal perpendicularly, the point R corresponds

transmissible in the direction NOr with velocity $\frac{1}{RN}$ and plane of risation perpendicular to ~~ON~~ RN.

. Generalisation suggests that, in the case of crystals belonging to a er type of general symmetry, there is a similar correspondence between ı ray and a point on an ellipsoid.

. Experiment confirms the rigorous accuracy of the generalisation.

. The surface of reference, whether a sphere, spheroid or ellipsoid, y be conveniently denoted by the term *optical indicatrix*.

. All the optical characters can be directly deduced from the icatrix itself, and reference to its polar reciprocal is for this purpose ecessary: further, it is possible to develop the characters from the sideration of *rays* alone.

0. The *front* of a pencil of rays which have started simultaneously m a point is part of the *ray-surface*; in the limit, if the pencil is of small ture and includes a given ray, the pencil-front is part of the plane ch touches the ray-surface where the ray meets it: hence the pencil- t corresponding to the given ray may be briefly designated as the *ray-* .

1. A plane passing through a ray and perpendicular to its plane of isation may be conveniently termed its *transverse* plane.

2. In such case, it follows that the normal to the ray-front corres- ding to the ray Or lies in the transverse plane $RNOr$ and is perpen- lar to OR, while the velocity of normal propagation of the front is asured by $\frac{1}{OR}$.

3. The normal RN is the direction of vibration of the ray corre- nding to the point R, if the most recent hypothesis as to the properties an elastic luminiferous ether is true.

4. The so-called primary and secondary optic axes are not axes of metry, nor even constant lines, of the crystal: they may with pre- on be denoted respectively as the *optic bi-normals* and *bi-radials*; for y are directions in which the two normals drawn from the centre to gent planes of the ray-surface having the same direction, or the two ii vectores of the ray-surface having the same direction, are respec- ly coincident with each other. A crystal may still be loosely termed *ral*, when it is merely desired to suggest that the interference-rings wn by a plate in convergent polarised light are rudely like those

which might be expected to be seen if the crystal had two axes, each id
tical in character with the optic axis of a tetragonal or hexagonal crys

15. By help of simple assumptions, which naturally present the
selves and are consistent with all known experimental results, Fresn
equation of the ray-surface may be deduced from the general princip
of undulations, without regard to the physical character of the perio
change.

On the Occurrence of Sapphire in Scotland.

By Professor M. F. Heddle, F.R.S.E.

[Read June 16th, 1891.]

T page 18 of Volume V. of the *Mineralogical Magazine* I gave an analysis of the red andalusite, which rarely occurs in quartzose veins ae schist of Clashnaree Hill, in Clova, Aberdeenshire.

ately, in preparing a section of a crystal of this mineral, which section cut at right angles to the main axis, I observed, after removing the ion from the last smoothing plate, a spot which reflected light with me brilliancy. Examination with the lens showed that this spot, h had an intense blue colour, was most perfectly polished, while the tal of andalusite which enclosed it hardly reflected light.

noticed in the paper which gives the analysis, the crystals of red dusite from this locality frequently have a central lozenge of a deeper somewhat purple tint.

he minute blue crystal or spot lay near to the line of meeting of the shades of red. It was surrounded by a narrow sheath of a white or yellow tint; and this sheath was markedly softer than even the dusite, having been removed below the mean level of the slice by the her emery.

he section had to be rubbed for some time with the very finest "float-y," before this sheath, the andalusite, and the blue crystal were ght nearly to a uniform level. It was then seen that the minute al was hexagonal, and that its centre was of a deeper blue than its substance.

he above seemed to leave little doubt that it was a sapphire, and the glance through the microscope, when the section was mounted, showed onjecture to be a correct one.

is a highly refractive, hexagonal crystal, of a pale blue tint, with a gonal deep blue centre, and an asteriated structure. Six rays of a almost as intense as is the centre, pass outwards nearly to the cir-erence.

Revolution, under crossed nicols, deepens the blue of the paler portic which colour changes to a reddish or brownish yellow on further rev tion.

The uniaxial system of rings is well seen. Some portions of the trian lar paler spaces are slightly biaxial.

The crystal is negative.

The diameter of the crystal of andalusite is about five-eighths of an in the central purplish lozenge occupies about one-third of its surface. diameter of the crystal of sapphire is about one-thirtieth of an inch.

The sheath which separates it from the andalusite is colourless, hig refractive, and formed of plates or thick fibres, which are laid on par to the six sides of the hexagon of sapphire.

From their position no rings can be expected.

As fibrolite of high lustre is associated with the andalusite, and as sheath was manifestly softer than the andalusite, it may consist of more siliceous variety of fibrolite termed xenolite, the formula of wh is $2Al_2O_3$, $3SiO_2$.

The process—or the progress of the smoothing of this section—sho in a very marked manner the extreme difference in the hardness minerals which are not far removed from each other in the numer scale. As the sheath is probably the most siliceous variety of fibrolite, hardness may be set down as 6. The red andalusite is very gem-l clear, and free from all inclusions; 7·5 must be assigned to it; 9 to sapphire. Yet at that stage of the smoothing at which the sapphire a polish like that of a diamond, the surface of the andalusite was pi all over, while to the lens the fibrolite looked like a dug-out trench.

In its higher grades the "the scale of hardness" is in no respect e approximately *relative*.

Probably, for the present, no better device is at our command tha ascertain, through the loss of weight, the amount abraded from a sla orthoclase or of rock crystal by equal bulks of equally commin powders of the substances whose hardness is to be determined.

It must be noted, however, that the conclusions to be drawn from a test would be to some extent vitiated by the toughness of such stances as chalcedony, jadeite, zircon and sapphire on the one hand—o the brittleness or facile cleavage of rock crystal, topaz, euclase and dian on the other.

On the Optic Properties of Gyrolite.

By Professor M. F. Heddle, F.R.S.E.

[Read June 16th, 1891.]

HE specimens of gyrolite which I obtained some summers ago at the Treshinish Islands have enabled me both to make sections for the roscope and to separate the pearly lustred plates of which the charac- stic orbicular masses are built up.

n both, but most perfectly in the latter, I have been able to get the e rings. This is, of course, only preliminary to the correct determina- of the crystalline faces, as promised by Professor Descloizeaux.

ll the first plates gave an apparently uniaxial figure. By employing liner plate I however found that the figure was biaxial, but that the i divergence was only 2° to 3°. On the outer side of each loop there pale but brilliant field of blue colour. Professor Descloizeaux had dy announced to me that the mineral was negative; this I corrobo-

he sections of orbicular gyrolite are among the most splendid of polar- pic objects, resembling a circularly-opened fan, of most brilliant and uptly contrasted colours.

he mutual cohesion of the plates is so feeble that only the smaller es—those in which in an ordinarily-sized section the rock forms the le circumference—can be utilised. Fortunately this mineral is in- ably in the centre of the druse-faröelite and mesolite, neither of which o fragile, underlying it.

Note on the Occurrence of Melanterite in the Upper Eocene Strata of Thames Basin.

By Rev. A. Irving, B.A., D.Sc.

Senior Science Master of Wellington College, Berks.

[Communicated by the Secretary. Read April 14th, 1891.]

UNDER the name of "Copperas" Bristow describes melanterite occurring, among other localities, in the Bagshot formation Dorsetshire (Glossary of Mineralogy). In Naumann-Zirkel's Mineralo (p. 491) it is stated that it seldom occurs distinctly crystallised in Natu Its derivation from iron pyrites, by the combined action of water a the atmospheric oxygen carried into the ground in solution in rain-wa is well known.

The purpose of the present note is to record its occurrence in the Up Eocene or Bagshot strata of the London Basin. Last winter a well dug near Wellington College Station on the South-Eastern Railway i the uppermost beds of the so-called Bracklesham or Middle Bagshot. this horizon beds of sand and clay, often constituting a very good "bri earth," when well mixed, are of common occurrence. In the pres instance thin bands of a very hard puce-coloured clay were interlamina with thin bands of rather coarse sand, which was coloured in many pla with a bright green. Grains of glauconite might be detected, but generally-diffused green colour of the sand-layers could not be attribu to the presence of that mineral; the colour, moreover, was a bottle-gr rather than the dark olive-green of glauconite. The beds in question w met with at a depth of 5 to 13 ft. from the surface of the ground, and c tained layers of lignite, a common accompaniment of FeS_2 in this for tion.

A considerable quantity of the material of the beds described reserved and exposed to atmospheric desiccation indoors. The result that in a few weeks the diffused green colour disappeared for the most p giving place to a rusty brown, as sulphate of iron is known to do w

osed to the air for a time, through the peroxidation of the base of the
This change at once suggested the presence of a native hydrated
hate of iron; and this proved to be the case, ordinary qualitative
ysis showing the presence of both iron and sulphuric acid.
partial quantitative analysis was then made of a portion of the sandy
erial collected from the small patches which still retained a distinctly
n colour. In order to avoid the risk of any loss of sulphuric acid from
substance by dry distillation it was thought best not to dry the sub-
ce at such a temperature as should secure the expulsion of all the water
re weighing the sample dealt with; preliminary weighing was therefore
ensed with. The sample was digested in excess of distilled water for a
or two, then filtered.
he *residue* from this was dried at 120° C. in an ordinary drying-
. and on being weighed gave 1·695 grams of a clean gritty sand,
a trace of argillaceous and carbonaceous material. The green tint had
ely gone even before it was dried.
the *filtrate* the sulphate was precipitated by $BaCl_2$, with the usual pre-
ions and 0·16 gram of $BaSO_4$ was obtained. By a simple calculation
equivalent weight of anhydrous $FeSO_4$ was found to be 0·104 gram.
ing now $7H_2O$ for the water of hydration of the native mineral, we get
gram of the hydrated salt as the equivalent of the 0·104 gram of
anhydrous salt. This gives 12·4 per cent. nearly as the proportion
e mineral melanterite in the sample examined.
he result is interesting, as throwing light in some cases upon the
in of the diffused green colour met with at various horizons in the
ls of this and other geological formations; and this must be the excuse
he appearance of the present short note.

(1.) *Tysonite and Bastnäsite;* (2.) *Meteoric Iron from Indian Va Township, Virginia;* (3.) *Anatase;* (4.) *Sapphire.*

By Geo. F. Kunz.

[Read June 16th, 1891.]

(1.) *Tysonite and Bastnäsite from Crystal Park, near Mainton Spri Colorado.*

ABOUT two years ago Mr. T. A. Herstand of Mainton Spri Colorado, made a remarkable find of tysonite and bastnäsite Crystal Park, near Mainton Springs, Colorado. The entire find consis of one crystalline cleavable mass of tysonite, covered with bastnäs weighing in all fully six kilograms. It was found in a coarsely crystal granite, and no other deposit has been discovered subsequently, altho considerable search has been made by various prospectors. Up to time of this find all the tysonite discovered in Colorado has not amou to more than 50 grams.

The tysonite was principally of a rich cinnamon-brown colour wit perfect basal cleavage, and frequently contained patches from one to th cm. in diameter which were entirely transparent.

In the fissures of both minerals is found a creamy white pulveru mineral which is evidently an alteration product of the bastnäsite.

Both the tysonite and bastnäsite exhibit very pronounced absorp spectra when viewed with a spectroscope; sections of both minerals parallel to the basal plane show the same uniaxial interference figure, w suggests that the two minerals may belong to the same crystalline tem, and that possibly the bastnäsite is either only an alteration pro of the tysonite, or a differently-coloured variety of the same mineral.

(2.) *An undescribed mass of Meteoric Iron from Indian Valley T ship, Floyd County, Virginia.*

This mass of meteoric iron was found in Indian Valley Townshi the spring of 1887, and weighs 14 kilos; it is a cubic iron, having a marked crystalline structure, and shows the Neumann lines on etc with dilute nitric acid.

furnished Mr. L. G. Eakins, of the U. S. Geological Survey, the following result on analysis :—

Iron	...	...	93·59
Nickel	...	...	5·56
Cobalt	...	...	·53
Copper	...	...	trace
Phosphorus	...	...	·27
Sulphur	...	...	·01
Silica	...	...	trace
			99·96

ne surface of the iron is very much corroded, and it is certainly not a nt fall.

.) *Octahedrite (Anatase) from near Placerville, Eldorado County, fornia.*

planted on some highly interesting quartz crystals with rather ual enclosures I observed some crystals of a mineral new to the fic Coast, and one which, though frequently observed in Europe, is f rather rare occurrence in North America—namely anatase.

to the present time only two specimens have come to hand; in both crystals were implanted on the quartz and were highly splendent; e specimen the crystals were a rich honey-brown by transmitted , on the other a deep metallic blue. In size none were longer than n. In form they showed the common habit of the Swiss crystals, ly the pyramid {111} surmounted by a more obtuse pyramid.

e anatase was found in a vein of quartz crystals, varying from the ht a finger up to masses weighing 80 or 90 lbs.

me of the quartz crystals were of more than usual interest—pellucid es weighing 50 lbs. each; others enclosed chlorite deposited in successayers; but the most interesting were those which enclosed, at a depth om three to ten mm. from the surface of the crystal, and arranged ding to the lines of growth, spherical masses (often silvery white), to five mm. in diameter, consisting of crystallised aggregates of ite or siderite. The crystalline form of the latter was rhombohedral, where exposed they had oxidised to a rusty-brown.

) *The Sapphire Deposits of the Northern Missouri River, near Helena, ana.*

was found, on a visit to this locality, that the sapphires are generally exagonal crystals without prism faces, or crystalline fragments.

They vary in colour from light blue, lavender, light red, light green, almost bottle green, and are very dichroic, appearing blue or green wh viewed across the prism, but pink or bright red when viewed along length of the crystal. They afford very brilliant gems with a rema able lustre; but no true (sapphire-) blue or true (ruby-) red crystals h been observed from this locality.

The greatest deposits are between Eldorado Bar, Ruby Bar, and ot bars along the Missouri River, for a distance of six miles, the cen point being 16 miles from Helena.

Until recent years they have not been systematically mined, but h been found in the sluices and pans in the mining for gold, with wh they are often found associated in a layer of fine glacial gravel immedia overlying the old slate rock.

In the deposit at Ruby Bar I observed, associated with the gold and gravel, a mastodon tusk about 3 ft. in length.

The associated minerals were rutile, topaz in brilliant white modi crystals resembling those from Durango, stream tin, curious concen masses of limonite pseudomorphous after pyrites, cyanite in broken c tals, chalcedony, calcite, and fine ruby-red garnet. From these the loca derives its name.

At Ruby Bar the sapphires were observed in a vein of eruptive r 6 ft. wide, cutting the green slate. The rock is very much altered, in it were found, associated with the sapphires, ruby red pyrope gar and sanidine felspars.

It is evident that all the sapphires found on the bars were derived f some such rock and were worn out during the glacial period.

Mr. H. Miers, of the Natural History Museum, London, reports that rock is a vesicular mica-augite-andesite.

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* Denotes an *Abstract* or *Review*.

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Sulphide replacing Stibnite and Orpiment 9
kel, F., on Cordierite 260

Errata in Vol. IX.

p. 262, line 18 from top,

for "*Krystalskelette*" *read* "*Krystallskelette.*"

p. 274, line 4 from bottom,

for "upper" *read* "right-hand."

p. 327, line 12 from bottom,

for "$lx'+my'+zz'$" *read* "$lx'+my'+nz'$."

p. 328, line 6 from bottom,

for "*Ot*" *read* "*Of.*"

p. 331, line 13 from top,

for "rays" *read* "front-normals."

p. 387, lines 1 and 2 from top,

for "$\frac{1}{ON}$" *read* "$\frac{1}{RN}$"

p. 391, line 12 from top,

for "thiner" *read* "thinner."